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Two-dimensional Quantum Field Theory, examples and applications

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Abstract

The main principles of two-dimensional quantum field theories, in particular two-dimensional QCD and gravity are reviewed. We study non-perturbative aspects of these theories which make them particularly valuable for testing ideas of four-dimensional quantum field theory. The dynamics of confinement and theta vacuum are explained by using the non-perturbative methods developed in two dimensions. We describe in detail how the effective action of string theory in non-critical dimensions can be represented by Liouville gravity. By comparing the helicity amplitudes in four-dimensional QCD to those of integrable self-dual Yang-Mills theory, we extract a four dimensional version of two dimensional integrability.

Contents

Chapter 1

General setup

As a natural extension of Quantum Mechanics, Relativistic Quantum Field Theory has demonstrated its predictive power in the calculation of processes in Quantum Electrodynamics. There are however conceptual and technical difficulties, since the local products of quantum fields, which are operator-valued distributions, are ill defined. This problem can only be resolved via the techniques of renormalization.

The general non-perturbative properties of quantum field theory were first extracted from a perturbative setup by the so-called LSZ formalism. Next, dispersion relations were found and were used to obtain non-perturbative information. These developments were followed by the axiomatic approach, known as constructive QFT, where functional analysis is the main technique. An important consequence of this approach was the PCT theorem connecting spin and statistics.

Meanwhile, all dynamical calculations in QFT were restricted to perturbation theory. This rendered calculations involving strong interactions unreliable. In addition information about the bound state spectrum could only be accessed within approximate non-perturbative - and often non-unitary–schemes. As a result, QFT fell into stagnation and was discredited for many years. These difficulties provided, in particular, a motivation for S-matrix theory. The predictive power of this theory turned out to be very small, since it was entirely based on kinematical principles, analyticity and the bootstrap idea. An underlying dynamical framework was lacking. Nevertheless, analyticity in the complex angular momentum plane led to the important concept of duality. An explicit realization of these concepts by the remarkable Veneziano formula led to a new parallel development, in the sixties, the dual models. However, the predictions of the dual models at high-energy scatterings were incorrect. Moreover, an analysis of the pole structure of higher-order corrections required the introduction of a somewhat mysterious new concept, the pomeron.

In the meantime, QFT had scored remarkable successes in the realm of the weak interactions. Moreover, symmetry principles had proven powerful in predicting the masses of strongly interacting particles without the recourse to dynamical calculations. This led to a revival of QFT in the late sixties. Consequently, in the seventies, much attention was given to the non-perturbative aspects. Quantum chromodynamics (QCD) was proposed as

the fundamental theory of the strong interactions. However, reliable non-perturbative calculations were still lacking in four dimensions and were only available for specific models in two-dimensional space-time[1]. It was understood that the short distance singularities of quantum field theory play a key role in the dynamical structure of the theory[9]. The experimental results on lepton-proton scattering at large momentum transfer, required that a realistic theory of the strong interactions be asymptotically free.

The first soluble model, describing the current-current interaction of massless fermions was discussed by Thirring[2] in 1958, as an example of a completely soluble quantum field theoretic model obeying the general principles of a QFT[3]. Subsequently, Schwinger[4] obtained an exact solution of Quantum Electrodynamics in 1+1 dimensions, QED_2. A number of interesting properties, such as the nontrivial vacuum structure of this model, were however only later revealed[5], and it was found that there is a long range Coulomb force for the charge sectors of the theory. This long range force was interpreted as being responsible for the confinement of quarks[6]. The problem of confinement and the related phenomenon of screening of charge quantum numbers in two dimensions have been extensively studied[7, 8], and have served as a basis for sharpening the similar concepts in higher dimensions. The surprisingly rich structure of two-dimensional quantum electrodynamics was found to describe several important features of non-abelian gauge theories, which were under investigation in the seventies.

Several further developments of increasing importance followed. Two-dimensional models which were exactly integrable classically were extensively studied[10]. Such integrable models are generally characterized by the existence of an infinite number of conservation laws[11]. In the cases where these conservation laws survive quantization, the S-matrices and their associated monodromy matrices can be computed exactly. Some of the results concerning classical integrability have also been generalized to higher dimensions[13].

In the framework of two-dimensional models, the possibility of writing fermions in terms of bosons (bosonization) has been a powerful method for obtaining non-perturbative informations. The building blocks of the bosonization scheme are the exponentials of the free bosonic fields. The fermion number of this composite operator is directly linked to the infrared behaviour of the zero mass scalar fields. This leads to a superselection rule[14], which makes the charged sectors appear in a very natural way.

The bosonization techniques become cumbersome when applied to non-abelian theories. Significant progress in the direction of non-abelian bosonization has been achieved[15, 16], and an equivalent bosonic action involving the principal sigma model plus a Wess-Zumino term was obtained.

Non-linear sigma models have a long history. A particularly important role has been played by the class of two-dimensional integrable non-linear sigma models, which have a geometrical origin[17]. They have been shown to share several properties with Yang-Mills theories in four dimensions[17, 18]. When quantized, non-linear sigma models also exhibit features, believed to be properties of realistic theories, such as dynamical mass generation, and a long range force[18] for simple gauge groups[20], disappearing after a suitable interaction with fermions (supersymmetric or minimal), which liberate the partons[23]. These

properties make them appealing as toy models for the strong interactions[19]. However, their geometrical origin makes them also very interesting mathematical objects to be studied in their own right, as well as in string theory.

Two-dimensional space-time has also proven to be an excellent laboratory for the study of gauge-anomalies and the consistency of anomalous chiral gauge theories. The exact solubility of two-dimensional chiral QED[21, 22] has played here an important role in opening up a whole new line of developments in the area of chiral gauge theories.

More recently, it has been shown that in two-dimensional quantum field theories, Poincaré and scale invariance imply the existence of an infinite dimensional symmetry group[24]. As a result, non trivial correlators can be exactly computed. They are found to be related to solutions of hypergeometric differential equations. The parameters labelling these equations, which are regarded as the critical indices, have been classified and characterize the correlators uniquely[24]. The above ideas may be generalized to include the interaction with conformally invariant gravity[25]. In the light-cone-gauge the theory simplifies dramatically, due to a new $SL(2,R)$ symmetry[25, 26]. The critical indices of the theory may be computed from a very simple equation relating them to the critical indices of the theory in flat space. The results have also been generalized to the supersymmetric case[27].

To summarize, two-dimensional models have been an extraordinary laboratory to test ideas in quantum field theory. The Thirring model provided a realization of an exactly soluble quantum field theory, while the Schwinger and the non-linear sigma models were found to exhibit properties of four dimensional non-abelian gauge theories. However, two-dimensional QFT also plays a direct role in the description of physical reality, having applications in string theories, as well as in statistical mechanics. In particular, the methods developed in two-dimensional QFT have been used to extract results concerning the critical behavior of models in statistical mechanics, using conformal invariance alone[24]. An extraordinary amount of physically interesting[28] as well as mathematically elegant concepts[29] have emerged from the study of such theories. There is a deep relationship between rational conformal invariance in two-dimensional space-time and the Chern-Simons action in three dimensions[30]. The Chern-Simons action is a key element in the generalization of the fermion-boson equivalence to three dimensional space-time[31], as well as playing an important role in the understanding of non-abelian anomalies of chiral gauge theories in any dimension. Beyond their status as a theoretical laboratory, and their applications in string theories and statistical mechanics, the study of these models has led to recent developments which opened new possibilities for applications of some of the above methods in the study of quantum field theories in higher dimensions. High-energy scattering amplitudes involving fields with definite helicity in four-dimensional Quantum Chromodynamics have a rather simple description, presumably related to integrable models[32], the same being true for high-energy scattering[33]. In the former case, the scattering amplitudes are related to solutions of self-dual Yang-Mills equation, while in the latter case the interaction of external particles is described by the two-dimensional Heisenberg Hamiltonian of spin systems.

Chapter 2

A survey of results

Here we restrict ourselves to the main results obtained in two-dimensional quantum field theory which had important consequences for realistic models. In subsequent chapters, the cases of more recent interest are reviewed in more detail, as well as recent applications to four dimensional quantum chromodynamics.

2.1 Schwinger model

The Schwinger model, Quantum Electrodynamics in two-dimensions, can be exactly solved in terms of the prepotentials for the gauge fields,

$$A_\mu = -\frac{\sqrt{\pi}}{e}\left(\tilde{\partial}_\mu \Sigma + \partial_\mu \tilde{\eta}\right) \ . \tag{2.1}$$

This, together with the bosonisation formulae,

$$J^\mu \equiv \overline{\psi}\gamma^\mu\psi = -\frac{1}{\sqrt{\pi}}\tilde{\partial}^\mu\Phi \quad , \quad J_5^\mu \equiv \overline{\psi}\gamma^\mu\gamma^5\psi = -\frac{1}{\sqrt{\pi}}\partial^\mu\Phi \ , \tag{2.2}$$

the field equations and the anomaly equation,

$$\partial_\mu J_5^\mu = \frac{e}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} \ , \tag{2.3}$$

leads to the conclusions that Φ and Σ differ by a massless field. That is to say, the current has a transverse part given in terms of the (physical) field Σ and a longitudinal part given in terms of the massless fields (which in turn have to be weakly zero). This condition defines the space of states. That is,

$$J^\mu = -\frac{1}{\sqrt{\pi}}\tilde{\partial}_\mu\Sigma + L_\mu \tag{2.4}$$

where

$$L_\mu = -\frac{1}{\sqrt{\pi}}\tilde{\partial}_\mu\varphi \ . \tag{2.5}$$

The Maxwell equation implies

$$\tilde{\partial}_\mu \left(\Box + \frac{e^2}{\pi} \right) \Sigma - \frac{e^2}{\sqrt{\pi}} L_\mu = 0 \ . \tag{2.6}$$

The only physical excitation of the theory is Σ, which is a free field of mass $\frac{e^2}{\pi}$, and the physical Hilbert space is characterized by requiring that the expectation value of L_μ vanishes. Observables are those operators which leave the Hilbert space invariant, that is

$$\begin{aligned} &\langle \psi' | L_\mu | \psi \rangle = 0 \\ &\mathcal{O} | \psi \rangle \in \mathcal{H}_{phys}, \quad \text{if } |\psi\rangle \in \mathcal{H}_{phys} \end{aligned}$$

Consequently, operators such as

$$\sigma_\alpha = e^{i\sqrt{\pi}\gamma^5_{\alpha\alpha}(\varphi+\eta) + i\sqrt{\pi}\int_{x^1}^\infty dz^1 \partial_0(\varphi+\eta)} \tag{2.7}$$

are observables, but act as constant operators, defining the θ vacuum structure of the model. The massless Schwinger model is characterized by the fact that physical fermion fields are not observables, but flavour quantum numbers are, since the force between distant quarks does not grow with distance. This fact characterizes screening. The massive model on the other hand, displays (with the exception of some θ worlds) a long-range force, linearly growing with distance, which means confinement.

2.2 Non-abelian generalization

It is desirable to generalize these results for two-dimensional quantum chromodynamics. For non-abelian gauge groups, bosonisation is more sophisticated: while in the abelian case fermions can be written in terms of bosons by means of an exponential map, in the non-abelian case such a map does not exist.

Bosonisation of two-dimensional QCD is achieved by writing the fermionic determinant in terms of a bosonic integration. Using the parametrization $eA_+ = U^{-1} i\partial_+ U$, $A_- = V i\partial_- V^{-1}$, the effective bosonic action $S_F[A, w]$ for the fermions is written in terms of the so-called gauged Wess-Zumino-Witten action,

$$\begin{aligned} S_F[A,g] = \Gamma[g] + \frac{1}{4\pi}\int \mathrm{d}^2x \, \mathrm{tr} \left[e^2 A^\mu A_\mu - e^2 A_+ g A_- g^{-1} - eiA_+ g\partial_- g^{-1} - eiA_- g^{-1}\partial_+ g \right] \quad , \\ S_F[A,g] \equiv \Gamma[UgV^{-1}] - \Gamma[UV] \end{aligned} \tag{2.8}$$

We thereby obtain the QCD_2 partition function

$$\mathcal{Z} = \int \mathcal{D}A\mathcal{D}g \mathrm{e}^{i(S_{YM} + S_F[A,g])} \quad .$$

Using algebraic identities, such as the Polyakov-Wiegmann relation

$$\Gamma[AB] = \Gamma[A] + \Gamma[B] + \frac{1}{4\pi} \int d^2x \mathrm{tr}[(A^{-1}\partial_+ A)(B\partial_- B^{-1})] \quad , \tag{2.9}$$

we arrive at the partition function for the full QCD_2, that is

$$Z = Z_F^{(0)} Z_{gh}^{(0)} \hat{Z}_{gh+}^{(0)} Z_{\tilde{W}'} Z_{\beta'} \quad , \tag{2.10}$$

where the first term represents free fermions, the second and third are the ghost terms, the fourth represents the negative metric excitations. These terms are all conformally invariant and thus describe the vacuum excitations. The last term in the above expression describes the massive excitations. However, all these fields are non-trivially connected by means of BRST constraints. Some of these constraints are derived from those already describing the original gauge fields, but there are constraints appearing from the various changes of variables which one uses to decouple the theory at the lagrangean level. Moreover, further complicated non-local constraints associated with the choice of a gauge-fixing condition (e.g., light-cone-gauge) also arise. These constraints, obtained by defining the currents

$$\begin{aligned} J_-(W) &= \frac{1}{4\pi} W i\partial_- W^{-1} \quad , \quad J_+(W) = \frac{1}{4\pi} W^{-1} i\partial_+ W \\ j_- &= \psi_1^{(0)} \psi_1^{(0)\dagger} + \{b_-^{(0)}, c_-^{(0)}\} \quad , \quad j_+ = \psi_2^{(0)} \psi_2^{(0)\dagger} + \{b_+^{(0)}, c_+^{(0)}\} \quad , \end{aligned}$$

are as follows

$$\begin{aligned} \Omega_+ &= -(1+c_V) J_+(\tilde{W}) + j_+ , \\ \Omega_- &= -(1+c_V) J_-(\tilde{W}') + j_- \quad , \\ \hat{\Omega}_- &= -\lambda^2 \beta (\partial_+^{-2} (\beta^{-1} i\partial_+ \beta)) \beta^{-1} + J_-(\beta) \\ & \quad -(1+c_V) J_-(\tilde{W}) + \{\hat{b}_-^{(0)}, \hat{c}_-^{(0)}\} \quad . \end{aligned}$$

The physical Hilbert space can be obtained in terms of these constraints. Notice that the fields β are not physical themselves. The β action itself describes an integrable model, but due to quantum corrections the β excitations are in principle not physical, while the physical mesons do not describe an integrable model, albeit they are almost stable against decay.[34]

The massive theory, although not factorizable (even only at the lagrangean level) can be rewritten in terms of fermions, and the question of screening can be discussed.

We understand the phenomenon with a pair of probe charges $(q, -q)$, a distance L apart, leading to the current

$$J^0 = q\,[\delta(x - L/2) - \delta(x + L/2)] = -\frac{e}{\sqrt{\pi}} \frac{\partial Q}{\partial x}$$

Suppose the colour is fixed, e.g. $q \sim \sigma_2 \in SU_2$.

The effective Lagrangian of the equivalent mechanical problem is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} E'^2 + \frac{1}{2} \frac{c_V + 1}{c_V} \Phi'^2 + 2\pi\lambda^2 E^2 - 2m^2 \cos 2\sqrt{\pi}(\Phi + E) + \\ & 2\sqrt{\pi}(c_V + 1)\lambda q E[\theta(x + L/2) - \theta(x - L/2)] \\ & -\frac{1}{2}(c_V + 1)^2 q^2 [\theta(x + L/2) - \theta(x - L/2)] - \frac{\psi'^2}{2c_V} \end{aligned}$$

where

$$\begin{array}{lll} \Phi{=}\varphi+\eta \quad , & \psi=\varphi+(c_V+1)\eta \quad , & \\ g{=}\mathrm{e}^{2i\sqrt{\pi}\varphi\sigma_2} \quad , & \beta=\mathrm{e}^{2i\sqrt{\pi}E\sigma_2} \quad , & \Sigma=\mathrm{e}^{2i\sqrt{\pi}\eta\sigma_2} \quad . \end{array}$$

The combinations

$$\chi_+ \approx E - a\Phi \quad \text{and} \quad \chi_- \approx \Phi - \epsilon a E \tag{2.11}$$

are described by the lagrangian

$$\mathcal{L} = -\frac{1}{2c_V}\psi'^2 + \sum_{\pm}\left[\frac{1}{2}\chi'^2_i + \frac{1}{2}m_i\chi_i^2 + \lambda Q_i\chi_i\right] \quad . \tag{2.12}$$

We see above that there are two parameters describing mass, and one expects already at this point a screening like potential.

Solving the classical problem (see Schwinger model) we obtain

$$V(L) = A\frac{1-\mathrm{e}^{-m_+L}}{m_+} + B\frac{1-\mathrm{e}^{-m_-L}}{m_-} \tag{2.13}$$

which always leads to screening.

Notice that $m_- \sim c_V$, and for the abelian case one always gets a confinement terms, unless $B = 0$.

2.3 Applications to four dimensional QCD

The issue of quasi-integrability, by which we mean a rather weakened version appears in four dimensional QCD. The case of high energy scattering and its relation to the Heisenberg model is already well know and sufficiently discussed.

Let us first review methods to deal with perturbative computations in QCD, a task whose difficulty is enhanced by the number of diagrams, colour indices, and polynomial vertices.

There are evidences for a weak version of integrability of the Yang-Mills theory. These resemble those of two-dimensional Yang-Mills theory, where integrability seems to fail by small deviations, when probed by means of the meson decay amplitudes.

Available methods for treating four dimensional QCD are:

• Helicity amplitudes

one deals with external particles with definite helicity. In the high energy, or massless fermion, limit one has maximal helicity violation.

• Colour decomposition

use of group theory to break amplitudes into gauge invariant pieces, with a fixed cyclic order of external legs. Useful for obtaining amplitudes which contain a large number of external gluons.

• Supersymmetry identities

although QCD is not supersymmetric, at tree level there being no fermion loops, the is no

loss considering the fermions in the adjoint representation. Thus, supersymmetry can be used to relate the amplitudes containing gluons to others containing scalars, which leads to significant simplifications. Usefull at one loop level.

• Recurrence relations

Amplitudes with one off-shell quark or gluon can be obtained using classical solutions of Yang-Mills equations. More specifically, knowing an amplitude with n external on-shell legs, one can use these relations to compute amplitudes with $(n+1)$ on-shell legs.

• String inspired methods

There are consistent string theories whose infinite-tension limit corresponds to a non-abelian gauge field. In such cases, one can do loop computations, by using the string formulation, as a bookkeeper of the algebra.

• Unitarity

use of Cutkovsky rules and analytic methods are also useful in simplifying the results.

Helicity amplitudes

Some results are extraordinarily simple, in spite of the difficulty of the subject.

Thus tree-level helicity amplitudes are such that if all helicities, or all but one, are the same, the amplitude vanishes. If only two are different, the tree amplitude will be simply:

$$\begin{aligned}\mathcal{A}(p_1,+,p_2,+,\cdots p_n,+)&=0\\ \mathcal{A}(p_1,-,p_2,+\cdots p_n,+)&=0\\ \mathcal{A}(p_1,-,p_2,-,\cdots p_n,+)&=ie^{n-2}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\end{aligned}$$

One loop results look like anomalies,

$$\begin{aligned}\mathcal{A}(p_1,+,p_2,+,p_3,+,p_4,+)&=i\frac{e^4}{2\pi^2}\frac{\langle 12\rangle^*\langle 34\rangle^*}{\langle 12\rangle\langle 34\rangle}\\ \mathcal{A}(p_1,-,p_2,+,p_3,+,p_4,+)&=i\frac{e^4}{2\pi^2}\frac{\langle 12\rangle\langle 34\rangle^*\langle 24\rangle^*}{\langle 12\rangle^*\langle 34\rangle\langle 24\rangle}\end{aligned}$$

Self-dual Yang-Mills theory

Self-dual Yang-Mills theory implies very simple results for helicity amplitudes. In fact, tree-level amplitudes with a fixed helicity can be given in terms of solutions of the self-dual Yang-Mills theory. It is described by

$$[\gamma^\mu,\gamma^\nu]\,F_{\mu\nu}=0\ ,$$

or

$$F_{0+3,0-3}=F_{1+i2,1-i2}\quad F_{0-3,1-i2}=0=F_{0+3,1+i2}$$

In the light-cone-gauge $A_{-}A_3 = 0$ we have a simple set of equations:

$$\begin{aligned}
G_{0+3,0-3} &= -\partial_{0-3}A_{0+3} \\
G_{1+i2,1-i2} &= \partial_{1+i2}A_{1-i2} - \partial_{1+i2}A_{1-i2} + i\,[A_{1+i2}, A_{1-i2}] \\
G_{0+3,1-i2} &= \partial_{0-3}A_{1-i2} \\
G_{0+3,1+i2} &= \partial_{0+3}A_{1+i2} - \partial_{1+i2}A_{0+3} + i\,[A_{0+3}, A_{1+i2}]
\end{aligned}$$

whose solution is given in terms of a single scalar field, namely

$$A_{1+i2} = \partial_{0-3}\Phi \quad , \qquad A_{0+3} = \partial_{1-i2}\Phi$$

where Φ obeys the equation

$$\partial^2\Phi + i\,[\partial_{1-i2}\Phi, \partial_{0-3}\Phi]$$

Such an equation of motion may be derived from the lagrangean

$$\begin{aligned}
\mathcal{L} &= \varphi\,[\Box\Phi + i\partial_{1-i2}\Phi, \partial_{0-3}\Phi] \\
&= \varphi\left[\Box\Phi + i\partial_{+}^{\dot\alpha}\Phi, \partial_{+\dot\alpha}\Phi\right]
\end{aligned}$$

The field φ has dimension 2 and cannot make an external leg except at tree level. Moreover, due to the dimension and helicity counting, all amplitudes are maximally helicity violating. Indeed, we find the previous amplitudes.

An iterative solution can be found in terms of the colour-ordered amplitude expansion. We have

$$\begin{aligned}
&\Phi \sim \phi(k_1)e^{-ik_1x}T^{a_1} + \\
&+i\sum \phi(k_1)e^{-ik_1x}T^{a_1}\cdots\phi(k_n)e^{-ik_nx}T^{a_n}(Q_1-Q_2)^{-1}(Q_2-Q_3)^{-1}\cdots(Q_{n-1}-Q_n)^{-1}
\end{aligned}$$

with $Q = k_{0+3}/k_{1-i2}$.

As noticed by Bardeen, the form of the solution corresponds to the (ordered) Bethe-Ansatz solutions found in two-dimensional integrable systems.

Note however: Coleman-Mandula theorem is a no-go for integrability in higher dimensional systems!!!

2.4 A note on conformal invariance

The use of conformal symmetry in quantum field theory has been advocated by Wess and Kastrup at the beginning of the sixties. Renewed interest in this topic resulted from Wilson's ideas about the short distance expansion of products of operator at nearby points, and the associated notion of anomalous dimensions of field operators, which are intimately linked to the high energy behaviour of renormalizable quantum field theories.

It was noted by Polyakov and others in the early seventies, that critical models implement a global conformal invariance which goes beyond pure scale invariance. This has led Polyakov to propose the use of conformal invariance as an essential ingredient in the study of the critical behaviour of statistical models at second order phase transitions.

Whereas scale transformations merely scale relative distances by a constant factor, conformal transformations involve a space-dependent factor. This imposes restrictions, which allow one to fix the two- and three-point functions at criticality. In general one is not able to go beyond that, since in general the conformal group is a finite dimensional Lie group, resulting in only a finite number of restrictions on the correlators.

In two dimensions the situation is drastically different, since here the conformal transformations are represented by all the analytic transformations in euclidean space. This will in fact enable us to reduce the two-dimensional problem to two one-dimensional ones. In fact, the significant restrictions imposed by the invariance under analytic transformations will ultimately lead to a classification of a large class of critical phenomena in two dimensions.

Conformal invariance also finds its application in string theory, two-dimensional gravity, as well as the non-linear sigma models.

2.4.1 The conformal group in two dimensions

In two dimensions the Killing-Cartan equation takes the form

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial \cdot \epsilon)\eta_{\mu\nu} \tag{2.14}$$

The diagonal elements of these equations then read $\partial_0\epsilon_0 + \partial_1\epsilon_1 = 0$, whereas for the off-diagonal elements we have $\partial_0\epsilon_1 + \partial_1\epsilon_0 = 0$. From these equations it follows that

$$(\partial_0 \pm \partial_1)(\epsilon_0 \pm \epsilon_1) = 0 \tag{2.15}$$

Hence in $D = 2$, a general conformal transformation is parametrized by $\epsilon_\pm = \epsilon_0 \pm \epsilon_1$, where $\epsilon_+(\epsilon_-)$ depends only on the light-cone-variable $x^+ = x^0 + x^1 (x^- = x^0 - x^1)$. The implications of this fact can best be appreciated by going to euclidean space via the usual substitution $x^0 = -ix_2^E$, $x^1 = x_1^E$, with the corresponding substitution $\epsilon^0 = -i\epsilon_2^E$, $\epsilon^1 = \epsilon_1^E$. The Cartan-Killing equations now read

$$\partial_\mu \epsilon_\nu^E + \partial_\nu \epsilon_\mu^E = (\partial^E \cdot \epsilon^E)\delta_{\mu\nu}$$

implying

$$\begin{aligned} \partial_1^E\epsilon_1^E - \partial_2^E\epsilon_2^E &= 0 \\ \partial_2^E\epsilon_1^E + \partial_1^E\epsilon_2^E &= 0 \end{aligned} \tag{2.16}$$

If we define the complex function

$$\epsilon = \epsilon_1^E + i\epsilon_2^E$$

and the complex variable

$$z = x_1^E + ix_2^E$$

then equations (2.16) are just the Cauchy-Riemann equations for the real and imaginary parts of the function ϵ,

$$\frac{\partial Re\epsilon}{\partial x_1^E} = \frac{\partial Im\epsilon}{\partial x_2^E}, \quad \frac{\partial Im\epsilon}{\partial x_1^E} = -\frac{\partial Re\epsilon}{\partial x_2^E} \quad . \tag{2.17}$$

This implies that the function ϵ depends only on the complex variable z, and ϵ^* on the complex variable $z^* = \overline{z}$:

$$\epsilon_1^E + i\epsilon_2^E = \epsilon(z); \quad \epsilon_1^E - i\epsilon_2^E = \overline{\epsilon}(\overline{z}) \tag{2.18}$$

Evidently, we have $\overline{\epsilon}(z^*) = \epsilon^*(\overline{z})$. The two dimensional conformal transformations thus coincide with the analytic transformations

$$z \longrightarrow f(z), \quad \bar{z} \longrightarrow \bar{f}(\bar{z}) \quad . \tag{2.19}$$

We have the following useful relations

$$\begin{aligned} \partial_z &= \frac{1}{2}(\partial_1^E - i\partial_2^E) \ , \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1^E + i\partial_2^E) \\ b \cdot x^E &= \frac{1}{2}(b\bar{z} + \bar{b}z) \quad b \cdot \partial^E = (b\partial_z + \bar{b}\partial_{\bar{z}}) \end{aligned}$$

where $b = b_1 + ib_2$ $\bar{b} = b_1 - ib_2$.

In the new coordinates the element of length and metric are respectively given by

$$ds^2 = dzd\bar{z} \ , \ \eta_{z\bar{z}} = \eta_{\bar{z}z} = \frac{1}{2} \ , \ \eta_{zz} = \eta_{\bar{z}\bar{z}} = 0 \quad . \tag{2.20}$$

Therefore, under a change of variable $z \rightarrow z'$ defined by $z = f(z')$,

$$ds^2 \rightarrow \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right) dzd\bar{z} \ . \tag{2.21}$$

Since $f(z)$ and $\bar{f}(z)$ are respectively analytic functions of z and antianalytic of $\bar{z}$, they have a Laurent expansion of the form

$$f(z) = z + \sum_{-\infty}^{\infty} \epsilon_n z^{n+1} \ , \quad \bar{f}(\bar{z}) = \bar{z} + \sum_{-\infty}^{\infty} \bar{\epsilon}_n \bar{z}^{n+1} \ . \tag{2.22}$$

We may regard this expansion as an expansion in the basis functions

$$\psi_n = z^{n+1}, \quad \bar{\psi}_n = \bar{z}^{n+1}$$

and write

$$z' = \left(1 + \sum_{-\infty}^{\infty} \epsilon_n L_n^c\right) z \quad \overline{z}' = \left(1 + \sum_{-\infty}^{\infty} \epsilon_n L_n^c\right) \bar{z}$$

where $L_n^c = z^{n+1}\frac{d}{dz}$, $\bar{L}_n^c = \overline{z}^{n+1}\frac{d}{d\bar{z}}$

The operators L_n^c and $\bar{L}_n^c$ are the generators of the analytic transformations and satisfy the loop algebra

$$\begin{aligned}
[L_n^c, L_m^c] &= -(n-m)L_{n+m}^c \ , \\
[\bar{L}_n^c, \bar{L}_m^c] &= -(n-m)\bar{L}_{n+m}^c \ , \\
[L_n^c, \bar{L}_m^c] &= 0 \ .
\end{aligned} \tag{2.23}$$

The superscript c is used to remind the reader that the L_n^c generate analytic transformations on classical functions. In the quantum case the first two commutators will be modified by an extension proportional to a central charge, while the third commutator remains unchanged. This defines the Virasoro algebra.

Since the L_n^c commute with the $\bar{L}_m^c$, the local conformal algebra is the direct sum $A \oplus \bar{A}$ of the two isomorphic algebras. This implies that the conformal group in two dimensions acts independently on z and $\bar{z}$. For this reason we may continue the Green functions of a conformal theory in $D = 2$ to a larger domain, where z and $\overline{z}$ are treated as independent variables, as already advertised. By taking $\overline{z}$ as the complex conjugate of z, we recover the original coordinates $(x_1, x_2) \in \mathrm{I\!R}^2$.

Chapter 3

Two-dimensional Quantum Chromodynamics

3.1 Introduction

The Schwinger model, or QED in two-dimensions with massless fermions, is exactly solvable. The physical space is generated by a single massive free boson, while the original fermions are bounded together by a screening force, such that the charge quantum number is not observable. In the massive fermion case, a long range force develops, and further quantum numbers associated with the fermion are permanently confined. The vacuum is degenerate, an issue which can be traced back to the break of chiral symmetry. A plethora of interesting physical results, expected to be true in a realistic parton model but which had never been firmly established, were proved[1].

Unlike the Schwinger model, quantum chromodynamics of massless fermions is no longer exactly solvable. It nevertheless serves as a very useful laboratory for studying problems such as the bound-state spectrum, algebraic structure and duality properties. These problems important tools for general understanding of realistic quantum field theories and are expected to be realized in four dimensional Quantum Chromodynamics. In particular, exact properties may be derived, by either using standard perturbation theory, or by obtaining its effective action using the heat kernel method. One arrives at an equivalent bosonic formulation in the form of a gauged Wess-Zumino-Witten (WZW) action.

The first attempt to obtain the particle spectrum dates back to 1974, and was based on the $1/N$ expansion[35], where N is the number of colours. In this limit one is led to a bound state spectrum corresponding asymptotically to a linearly rising Regge trajectory. The use of the principal-value prescription in dealing with the infrared divergencies is however highly ambiguous due to the non-commutative nature of principal-value integrals. Moreover, the result for the fermion propagator is tachyonic for a small fermion mass as compared with the coupling constant, hence, in particular, for $m = 0$. This has made 't Hooft's solution a controversial issue[36].

In the large N approximation the gluons remain massless, since fermion loops do not

contribute to the Feynman amplitudes. This is unlike the $U(1)$ case, where the photon acquires a mass via an intrinsic Higgs mechanism. This has led to the speculations that QCD_2 may in fact exist in two phases associated with the weak and strong coupling regimes. In this picture, the large N limit would correspond to the weak-coupling limit ('t Hooft's phase), with massless gluons and a mesonic spectrum described by a Regge trajectory. In such a case, the Regge behaviour of the mesonic spectrum is compatible with confinement. In the strong coupling regime (Higgs phase), on the other hand, the gluons would be massive, and the original $SU(N)_c$-symmetry would be broken down to the maximal abelian subgroup (torus) of $SU(N)_c$.

The behaviour of the theory in the strong or weak coupling limits is rather subtle. The theory is asymptotically free, as it should since it is super-renormalizable. In the strong coupling limit, it is expected to be in the confining phase. Indeed, in the infinite infrared cut-off limit quarks disappear from the spectrum, which consists of mesons lying approximately on a Regge trajectory. One obtains a simple and finite solution for the fermion self-energy (SE), and the fermion two-point function[37]. This solution is useful for analyzing properties related to the high-energy scattering amplitudes.

Functional techniques are also very powerful for arriving at equivalent bosonic actions for the fermionic theory, and for obtaining a non-abelian bosonization formulae.

In the framework of conventional perturbation theory, by formally summing the perturbation series, we arrive at exact representations for the gauge current, the determinant of the Dirac operator and the Dirac Green's functions in an external field. The external field current $J_\mu(x \mid A)$ and fermionic determinant $\det i \not{D}$ of a $U(1)$ gauge theory in two dimensions (QED_2) are exactly calculable. In fact they are, respectively, functionals of first and second degree in A_μ, reflecting the fact, that the perturbation series for $J_\mu(x \mid A)$ and $\ln\det\ i \not{D}[A]$ (connected one-loop graphs) terminates at the first non-trivial order in the coupling constant e. Subsequently, we could also solve for the exact Dirac-Greens functions in this case. Moreover, the structure of both the determinant and the Greens function is such that all correlation functions of QED_2 can be constructed explicitly via the generating functional for the massless case.

The same is not true for QCD_2, where the perturbation series for $\ln\det i\not{D}$, corresponding to the effective action and $J^a_\mu(x \mid A)$ no longer terminates. Nevertheless, the effective action may still be calculated non-perturbatively in closed form by the Fujikawa method, or the heat-kernel (proper-time) method, by integrating the anomaly equation, or finally by summing the perturbative series. The result can be expressed in terms of a gauge invariant combination of the gauge potentials in the form of a Wess-Zumino-Witten action.

We shall primarily be dealing with massless fermions. The massive fermion case can in general not be dealt with, if one seeks exact results. It is however possible to compute the functional determinant as an expansion in the inverse of the mass. The problem of screening and confinement can however be analysed along the lines of the $U(1)$ case. However, unlike the $U(1)$ case, the screening phase prevails in the non-abelian theory[38, 39].

3.2 The Wess-Zumino-Witten theory

That a bosonic action can be used to describe baryons was foreseen long time ago[40]: it has been known that the baryon number of the solution of a certain bosonic theory is non-zero. Moreover, Lagrangeans of the type of a sigma model containing a topological term provided a rather reasonable description of low energy hadron physics.

In two-dimensional space-time, we aim at an equivalent bosonic action for massless fermions, transforming under a non abelian symmetry group. One is thus led to consider the two dimensional action

$$\Gamma[g] \equiv S_{WZW} = S_{P\sigma M} + S' \quad , \tag{3.1}$$

where

$$S' = nS_{WZ} = \frac{n}{4\pi}\int_0^1 dr \int d^2x \epsilon^{\mu\nu} tr g_r^{-1}\partial_r g_r g_r^{-1}\partial_\mu g_r g_r^{-1}\partial_\nu g_r \tag{3.2}$$

and $g_r(r,x)$ extends $g(x)$ in such a way that $g_r(0,x) = 1$ and $g_r(1,x) = g(x)$. The principal sigma model action reads

$$S_{P\sigma M} = \frac{1}{2\lambda^2}\int d^2x tr\, \partial^\mu g^{-1}\partial_\mu g \quad . \tag{3.3}$$

We shall refer to S_{WZ}, $S_{P\sigma M}$ and S_{WZW} as the Wess-Zumino (WZ), principal sigma model ($P\sigma M$), and WessZuminoWitten (WZW) action, respectively[16].

Existence of a Critical Point

Consider the one parameter family of actions (3.1), to which shall refer as the WessZumino-NovikovWitten (WZNW) action. For the choice $\frac{4\pi}{\lambda^2} = n$, S_{WZNW} is the equivalent bosonic action of the (conformally invariant) theory of N non-interacting massless fermions. To demonstrate this we examine the equations of motions and one-loop β-function corresponding to (3.1).

The equations of motion are obtained by computing the functional variation of S_{WZNW} with respect to g_{ij}. From (3.1-3.3) we obtain two alternative forms for δS_{WZNW} (see [41] for details):

$$\delta S_{WZNW} = \int d^2x \,\mathrm{tr}\, g^{-1}\delta g \left(\frac{1}{\lambda^2}g^{\mu\nu} - \frac{n}{4\pi}\epsilon^{\mu\nu}\right)\partial_\mu(g^{-1}\partial_\nu g) \quad , \tag{3.4}$$

$$= \int d^2x tr \delta g g^{-1}\left(-\frac{1}{\lambda^2}g^{\mu\nu} - \frac{n}{4\pi}\epsilon^{\mu\nu}\right)\partial_\mu\left(g\partial_\nu g^{-1}\right) \quad . \tag{3.5}$$

where the integration over r could be performed.

Setting $\delta S_{WZNW} = 0$, we obtain as equations of motion the conservation laws,

$$\partial_\mu\left(\frac{1}{\lambda^2}g^{\mu\nu} - \frac{n}{4\pi}\epsilon^{\mu\nu}\right)(g^{-1}\partial_\nu g) = 0 \tag{3.6}$$

$$\partial_\mu\left(\frac{1}{\lambda^2}g^{\mu\nu} + \frac{n}{4\pi}\epsilon^{\mu\nu}\right)(g\partial_\nu g^{-1}) = 0 \quad . \tag{3.7}$$

For the choice

$$\frac{4\pi}{\lambda^2} = n \quad , \tag{3.8}$$

Eqs. (3.6-3.7) just express the conservation of the left (L) and right (R) moving currents

$$j^\mu_L(x) = -i\frac{n}{4\pi}(g^{\mu\nu} - \epsilon^{\mu\nu})(g^{-1}\partial_\nu g) \quad , \quad j^\mu_R(x) = -i\frac{n}{4\pi}(g^{\mu\nu} + \epsilon^{\mu\nu})(g\partial_\nu g^{-1}) \quad . \tag{3.9}$$

$$j_+(x) = -\frac{in}{2\pi}g^{-1}\partial_+ g = i2\hat{\Pi}^T g - \frac{in}{2\pi}g^{-1}\partial_1 g \quad , \tag{3.10}$$

$$j_-(x) = -\frac{in}{2\pi}g\partial_- g^{-1} = -i2g\hat{\Pi}^T + \frac{in}{2\pi}g\partial_1 g^{-1} \quad , \tag{3.11}$$

This indicates that at the critical point the WZNW action is a natural candidate for describing N non-interacting massless fermions.

3.2.1 Properties at the Critical Point: the Kac Moody algebra

From the above discussion, we expect the theory to have a conformally invariant fixed point. To investigate the properties of the WZNW action (at the critical point) further, we proceed with a canonical quantization of the theory.

The WZ action depends linearly on the time derivative of g. Hence it proves useful to write it in the form[41]

$$S'[g] = \frac{n}{4\pi}\int d^2x \, tr A(g)\partial_0 g \quad , \tag{3.12}$$

where we have formally integrated over r, and $A(g)$ is a matrix-valued function of g and $\partial_1 g$. We thus obtain for the momentum conjugate to g_{ij}

$$\Pi_{ij} = \frac{n}{4\pi}\partial_0 g^{-1}_{ji} + \frac{n}{4\pi}A_{ji} \equiv \hat{\Pi}_{ij} + \frac{n}{4\pi}A_{ji} \quad . \tag{3.13}$$

Hence there are no constraints, and we can compute the Poisson Brackets, obtaining

$$\left\{j^a_+(x), j^b_+(y)\right\} = 2f^{abc}j^c_+(x)\delta(x^+ - y^+) + \frac{2n}{\pi}\delta^{ab}\delta'(x^+ - y^+) \tag{3.14}$$

$$\left\{j^a_-(x), j^b_-(y)\right\} = 2f^{abc}j^c_-(x)\delta(x^- y^-) + \frac{2n}{\pi}\delta^{ab}\delta'(x^- y^-) \tag{3.15}$$

$$\left\{j^a_+(x), j^b_-(y)\right\} = 0 \quad . \tag{3.16}$$

with $j^a = \mathrm{tr}\, j\tau^a$, and τ^a the $SU(N)$ generators. Here "prime" on δ represents the derivative with respect to the argument of δ. Thus, we have two Kac-Moody algebras with central extension $k = n$.

It is easy to show that (3.14-3.16) defines also the algebra one obtains in a field theory of N free fermions, with the action $S = \frac{i}{2}\int d^2x\overline{\psi}^i \not{\partial}\psi^i$.

Since a Kac-Moody algebra defines a conformally invariant theory uniquely for $k = \pm 1$ (unitary irreducible representations are unique), we are led to identify the currents (3.9-3.11)

with $\mathrm{tr}(\tau^a j_\pm)$. To complete this picture, we need to identify both the fermionic representation of the bosonic field g_{ij}, and the energy-momentum tensor in these theories.

It is clear that the WZW action corresponds to an interacting conformally invariant field theory for general values of k. It can be used in the bosonisation procedure for higher representations of interacting fermions, as in the case of QCD_2.

We can now prove the following important result due to Polyakov-Wiegman[15].

Theorem

$$\Gamma[AB] = \Gamma[A] + \Gamma[B] + \frac{1}{4\pi}\int d^2x(g_{\mu\nu} + \epsilon_{\mu\nu})\mathrm{tr}(A^{-1}\partial_\mu A)(B\partial_\nu B^{-1}). \tag{3.17}$$

Proof: It is convenient to use the notation

$$\Gamma[G] = S_{P\sigma M}[G] + S_{WZ}[G] \quad , \tag{3.18}$$

where $S_{P\sigma M}$ is the principal sigma model action, and the functional $S_{WZ}[G]$ is the Wess-Zumino action

Replacing $G_r(G)$ by $A_rB_r(AB)$ and using the cyclic properties of the trace one readily finds

$$S_{P\sigma M}[AB] = S_{P\sigma M}[A] + S_{P\sigma M}[B] + \frac{1}{4\pi}\int d^2x\mathrm{tr}[(A^{-1}\partial_\nu A)(B\partial_\mu B^{-1})]. \tag{3.19}$$

The corresponding calculation for the Wess-Zumino term is more involved. Using $(AB)^{-1}\partial(AB) = B^{-1}[A^{-1}\partial A + (\partial B)B^{-1}]B$, one finds after a lengthy calculation, that

$$S_{WZ}[AB] = S_{WZ}[A] + S_{WZ}[B] - \frac{1}{4\pi}\int_0^1 dr \int d^2x\epsilon_{\mu\nu}\mathcal{W}_{\mu\nu} \quad , \tag{3.20}$$

where $\mathcal{W}_{\mu\nu}$ can be written in the form

$$\mathcal{W}_{\mu\nu} = \frac{d}{dr}\mathrm{tr}[(A^{-1}\partial_\mu A)(B\partial_\nu B^{-1})] - \partial_\mu[(B\partial_\nu B^{-1})A^{-1}\dot{A}] - \partial_\nu[(A^{-1}\partial_\mu A)B\dot{B}^{-1}] \quad . \tag{3.21}$$

The second and third term in (3.21) contribute a surface term to $S_{WZ}[AB]$, which we drop. The first term gives a contribution which can be trivially integrated in r. Hence

$$\Gamma[AB] = \Gamma[A] + \Gamma[B] + \frac{1}{4\pi}\int d^2x\mathrm{tr}[(A^{-1}\partial_+ A)(B\partial_- B^{-1})] \quad , \tag{3.22}$$

which is the Polyakov-Wiegmann identity.

3.3 QCD_2:currents and functional determinant.

Quantum chromodynamics is defined by the Lagrange density

$$\mathcal{L} = -\frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \overline{\psi}_i(i\,\partial\!\!\!/ + eA\!\!\!/)\psi_i \quad , \tag{3.23}$$

where $F^{\mu\nu}$ is the chromoelectric field tensor, and the fermions ψ_i are in the fundamental representation of the gauge group.

The current $J^a_\mu = \overline{\psi}\gamma_\mu \tau^a \psi$ is covariantly conserved as a consequence of the Dirac equation. For a gauge-invariant regularization such a conservation law holds after quantization. We consider in general an external-field current, which is obtained by differentiating the functional

$$W[A] = -i \ln \frac{\det i \not{D}[A]}{\det i \not{\partial}} \quad , \tag{3.24}$$

with respect to A^μ_a, and is given by the expression

$$eJ^a_\mu(x|A) = \frac{\delta W}{\delta A^\mu_a(x)} \quad . \tag{3.25}$$

The functional $W[A]$ represents an effective action for the gauge field A_μ.

In 1+1 dimensions the fermionic part of the Lagrangian (3.23) is classically invariant under both $U(1)$ and chiral gauge transformations. For a $U(1)$ gauge invariant regularization, the local chiral symmetry is broken at the quantum level. Following Fujikawa, we can view this fact as the non-invariance of the fermionic measure under a local chiral change of variables. The corresponding Jacobian is obtained from the anomalous behaviour of the effective action under this transformation. With the definition of the axial vector current depending on the external gauge field as $J^b_{5\mu}(x|A) = \epsilon_{\mu\nu} J^{\nu b}(x|A)$, one thereby obtains from (3.25) the anomaly equation

$$\mathcal{D}^{ab}_\mu J^{5\mu b} = -\widetilde{\mathcal{D}}^{ab}_\mu J^{\mu b} = \frac{e}{2\pi}\epsilon_{\mu\nu} F^{\mu\nu a} \quad . \tag{3.26}$$

where $\mathcal{D}$ is the gauge covariant derivative in the adjoint representation.

The above anomaly equation leads to the pseudo-divergence of the Maxwell equation, that is,

$$\epsilon_{\nu\rho}\mathcal{D}^\rho \mathcal{D}_\mu F^{\mu\nu} + e\widetilde{\mathcal{D}}_\mu J^\mu = -\frac{1}{2}\left(\mathcal{D}^2 + \frac{e^2}{\pi}\right)\epsilon_{\mu\nu}F^{\mu\nu} = 0 \quad , \tag{3.27}$$

showing a mass generation for the gauge field, analogous to the one in the Schwinger model.

Furthermore, it is possible to compute the external field current $J^a_\mu(x|A)$ by integrating (3.26). Using the kernel $K^{ab}_\mu(x,y|A)$ defined by the equations

$$\mathcal{D}^{ab}_\mp K^{bc}_\pm = \mp \delta^{ac}\delta(x-y) \quad , \tag{3.28}$$

we have

$$J^a_\pm = \frac{e}{2\pi}\int \mathrm{d}^2 y\, K^{ab}_\pm(x,y|A)\epsilon_{\rho\sigma}F^{\rho\sigma b}(y) \quad . \tag{3.29}$$

Defining the two-point functions $D_\pm = -\frac{i}{4\pi x^\mp}$ we may represent K_μ as a power series expansion in terms of the gauge field in the fundamental representation, $A_\mu = \sum_c A^c_\mu \tau^c$. One

finds

$$\begin{aligned} K^{ab}_{\pm}(x,y|A)=&\delta^{ab}D_{\pm}(x-y) - \\ &-\sum_{n=1}^{\infty}(-e)^n\int \mathrm{d}^2x_1\cdots\mathrm{d}^2x_n\, D_{\pm}(x-x_1)\cdots D_{\pm}(x-x_n) \\ &\times\mathrm{tr}\left\{\tau^a\left[A_{\mp}(x_1),[A_{\mp}(x_2),\cdots[A_{\mp}(x_n),\tau^b]]\cdots\right]\right\} \quad . \end{aligned} \tag{3.30}$$

Substituting (3.30) into Eq. (3.29) and making use of the fact that the gauge field strength F_{+-} may be alternatively written in the form $F_{+-} = -\partial_- A_+ + \mathcal{D}_+ A_-$ or $F_{+-} = -\mathcal{D}_- A_+ + \partial_+ A_-$, one obtains, after a partial integration (and use of (3.28),

$$\begin{aligned} J^a_{\pm}(x|A)=&\frac{e}{2\pi}A_{\pm}(x) - \frac{e}{2\pi}\int \mathrm{d}^2y\, K^{ab}_{\pm}(x,y|A)\partial_{\pm}A^b_{\mp} \\ =&\frac{e}{2\pi}\left[A_{\pm} - \int \mathrm{d}^2y\,\partial_{\pm}D_{\pm}(x-y)A^a_{\mp}(y)\right] \\ +&\frac{i}{2\pi}\sum_{n=2}^{\infty}(-ie)^n\int \mathrm{d}^2x_1\cdots\mathrm{d}^2x_n\, D_{\pm}(x-x_1)\cdots D_{\pm}(x-x_n) \\ \times&\mathrm{tr}\left\{\tau^a\left[A_{\mp}(x_1),[\cdots[A_{\mp}(x_{n-1}),\partial_{\pm}A_{\mp}(x_n)]]\cdots\right]\right\} \quad . \end{aligned} \tag{3.31}$$

From (3.31) we now compute the effective action by integrating (3.25)

$$\begin{aligned} W[A]=&W[0] - \frac{ie^2}{2\pi}\int \mathrm{d}^2x\,\delta^{ab}A^a_{\mu}\left(g^{\mu\nu} - \frac{\partial^{\mu}\partial^{\nu}}{\partial^2}\right)A^b_{\nu}(x) \\ &+\frac{i}{2}\sum_{n=2}^{\infty}\frac{(ie)^{n+1}}{n+1}\int \mathrm{d}^2x\ \mathrm{tr}\left[A_-(x)T^{(n)}_+(x|A) + A_+(x)T^{(n)}_-(x|A)\right] \quad , \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} T^{(n)}_{\pm}(x|A)=&-\frac{1}{2\pi}(-1)^n\int \mathrm{d}^2x_1\cdots\mathrm{d}^2x_n\, D_{\pm}(x-x_1)\cdots D_{\pm}(x-x_n) \\ &\times[A_{\mp}(x_1),[\cdots[A_{\mp}(x_{n-1}),\partial_{\pm}A_{\mp}(x_n)]]\cdots] \quad . \end{aligned} \tag{3.33}$$

The exact summation of the series can be performed. We first define the generating functional

$$\hat{T}_{\pm}(x,t|A) = \sum_{2}^{\infty}\frac{(iet)^{n-1}}{(n-1)!}T^{(n)}_{\pm}(x|A) \tag{3.34}$$

such that

$$T_{\pm}(x|A) = ie\int_0^1 dt T_{\pm}(x,t|A)\ . \tag{3.35}$$

Further we note that the functional $\hat{T}_{\pm}(x,t|A)$ satisfies the differential equation

$$\partial_{\mp}T_{\pm} = -\frac{e}{8\pi}\partial_{\pm}A_{\mp} + ie[A_{\mp},T_{\pm}] \tag{3.36}$$

with $A_\pm(x,t) = tA_\pm(x)$. The above equation is easily solved by the expression

$$\hat{T}_\pm = \frac{i}{8\pi}\partial_\pm \hat{U}_\pm \hat{U}_\pm^{-1}(x,t) \tag{3.37}$$

where $\partial_\mp U_\pm = ieA_\mp U_\pm$. Replacing the gauge field by the U fields defined above in the expression for W (3.32), we find

$$\begin{aligned} W[A] =& \frac{1}{4\pi}\int_0^1 dt \int d^2x \,\mathrm{tr}\, \partial_+ U_+ U_+^{-1} \partial_t(\partial_- U_+ U_+^{-1}) + \\ &+ \frac{1}{4\pi}\int_0^1 dt \int d^2x \,\mathrm{tr}\, \partial_+ U_- U_-^{-1} \partial_t(\partial_- U_- U_-^{-1}) + \frac{1}{4\pi}\int d^2x \,\mathrm{tr}\, A_+ A_- \quad . \end{aligned} \tag{3.38}$$

The first and second terms are of exactly the same form. The significance of these terms can be understood by writing the first term as a sum of the symmetric plus anti-symmetric combinations in the $+$ and $-$ variables. The symmetric part is readily seen to be a total derivative in the t variable, and equals the principal sigma model action for a matrix-valued field. The anti-symmetric part is handled by applying the t derivative. When it acts on U^{-1}, the two terms obtained are exactly the topological term in the WZW action. When it acts on the derivative term, one can see that a total derivative is obtained, that is $\partial_+[\partial_- U^{-1}\partial_t U] - \partial_-[\partial_+ U^{-1}\partial_t U]$. Therefore we obtain the sum of the WZW action for U_+, the corresponding one for U_-, and the contact term in A_+A_-, which according to eq.(3.22) leads to the WZW action for the product U_+U_-, which is the gauge invariant combination. A similar result is obtained using other techniques[1].

The gauged WZW action

As already remarked, the fermionic part of the Lagrangian (3.23) is invariant under local gauge transformations $SU(N)$, as well as $SU(N)_L \times SU(N)_R$, for both right (R) and left (L) components, that is,

$$\begin{aligned} \psi_{\substack{R\\L}} &\rightarrow w_{\substack{R\\L}} \psi_{\substack{R\\L}} \quad , \\ A_\pm &\rightarrow w_{\substack{R\\L}} \left(A_\pm + \frac{i}{e}\partial_\pm\right) w_{\substack{R\\L}}^{-1} \quad . \end{aligned} \tag{3.39}$$

This transformation corresponds to pure vector gauge transformation when $w_R = w_L = w$, while when $w_R = w_L^{-1} = w$ it corresponds to a pure axial vector transformation. If we use a change of variables parametrizing $A_\pm$ as

$$eA_+ = U^{-1} i\partial_+ U \quad , \quad A_- = V i\partial_- V^{-1} \tag{3.40}$$

the transformations corresponding to axial transformations reduce to $U \to Uw^{-1}$ and $V \to w^{-1}V$.

The above transformation is not a symmetry of the effective action $W[A]$ due to the axial anomaly. This non-invariance may used in order to express the fermionic functional determinant in terms of a new bosonic action $S_F[A,w]$ for the fermions defined by

$$S_F[A,g] \equiv \Gamma[UgV^{-1}] - \Gamma[UV] \quad . \tag{3.41}$$

Using the invariance of the Haar measure, we evidently have, up to an irrelevant constant,

$$\det i\,\not{D} \equiv \mathrm{e}^{iW[A]} = \int \mathcal{D}g\,\mathrm{e}^{iS_F[A,g]} \quad . \tag{3.42}$$

Thus $S_F(A,g)$ plays the role of an equivalent bosonic action for fermions minimally coupled to gauge fields. Its explicit form may be obtained by repeated use of the Polyakov-Wiegmann formula (3.22), and we obtain

$$S_F[A,g] = \Gamma[g] + \frac{1}{4\pi}\int \mathrm{d}^2x\,\mathrm{tr}\,\left[e^2A^\mu A_\mu - e^2A_+gA_-g^{-1} - eiA_+g\partial_-g^{-1} - eiA_-g^{-1}\partial_+g\right] \quad , \tag{3.43}$$

which represents $\det i\not{D}$ in terms of bosonic degrees of freedom. Hence we have arrived at a representation of the highly non-trivial functional $\det i\not{D}[A]$ in terms of a simple effective action, at the expense of introducing as additional bosonic group-valued field g, over which one has to integrate. As seen from (3.43), one recovers the equivalent bosonic action of free fermions in the limit $e \to 0$, as expected. This circumvents the problem of having to replace the measure $\mathcal{D}A$ by $\mathcal{D}U\mathcal{D}V$. In this way, we obtain for the QCD_2 partition function

$$\int \mathcal{D}A\mathcal{D}g\mathrm{e}^{i(S_{YM}+S_F[A,g])} \tag{3.44}$$

where $\mathcal{D}A$ stands for the measure and includes the gauge fixing factor.

In the $U(1)$ case, where the Wess-Zumino term in (3.2) vanishes, only the principal σ-model is left, that is,

$$S = -\frac{1}{8\pi}\int \mathrm{d}^2x\,(\partial_\mu\Sigma^{-1})(\partial_\mu\Sigma) \quad , \tag{3.45}$$

where $\Sigma \equiv UV$. In the $U(1)$ case we consider the parametrization

$$U = e^{-i\sqrt{\pi}(\varphi+\phi)} \quad , \quad V = e^{i\sqrt{\pi}(\varphi-\phi)} \quad , \quad \Sigma = \mathrm{e}^{-i2\sqrt{\pi}\phi} \quad , \tag{3.46}$$

leading to

$$A_+ = \sqrt{\pi}\partial_+(\varphi+\phi) \quad , \quad A_- = \sqrt{\pi}\partial_-(\varphi-\phi) \quad , \quad F_{\mu\nu} = \epsilon_{\mu\nu}\sqrt{\pi}\Box\phi \quad . \tag{3.47}$$

In terms of the ϕ field the action reads

$$S = \int d^2x[-\frac{1}{2}\,(\partial_\mu\phi)^2 + \frac{\pi}{2e^2}\,(\Box\phi)^2] \quad , \tag{3.48}$$

where the last term corresponds to the free action for the gauge field.

Introducing unit in the corresponding partition function, adding to the action the (quadratic) term

$$\delta S = -\frac{e^2}{2\pi}\int d^2x\left(E + \frac{\pi}{e^2}\Box\phi\right)^2 \tag{3.49}$$

and integrating over the auxiliary field E, we find for the action

$$S = \int d^2x \left[-\frac{1}{2} (\partial_\mu \phi)^2 - \frac{e^2}{2\pi} E^2 + \partial_\mu E \partial^\mu \phi \right] \tag{3.50}$$

$$= \int d^2x \left[-\frac{1}{2} (\partial_\mu (\phi - E))^2 + \frac{1}{2} (\partial_\mu E)^2 - \frac{e^2}{2\pi} E^2 \right] \tag{3.51}$$

from which we see that the field $\eta = \phi - E$ is massless, has negative metric and decouples, realizing the well known negative metric excitations of the Schwinger model, while E represents the usual meson of mass $e/\sqrt{\pi}$.

3.4 Decoupling the dynamics.

3.4.1 QCD$_2$ in the local decoupled formulation and BRST constraints

The partition function of two-dimensional QCD in the fermionic formulation (before gauge fixing) is given by the expression

$$Z = \int \mathcal{D}A_+ \mathcal{D}A_- \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS[A,\psi,\bar{\psi}]} \tag{3.52}$$

with the action

$$\mathcal{S}[A, \psi, \bar{\psi}] = \int d^2x \left[-\frac{1}{4} \operatorname{tr} F^{\mu\nu} F_{\mu\nu} + \psi_1^\dagger (i\partial_+ + eA_+)\psi_1 + \psi_2^\dagger (i\partial_- + eA_-)\psi_2 \right]. \tag{3.53}$$

Our aim is to obtain a bosonic formulation of the theory, in such a way that structural relations, hidden in the fermionic formulation are made clearer in the bosonic counterpart. We first make some useful change of variables, obtaining a formulation in terms of matrix-valued fields which decouple at the partition function level, but which are not totally decoupled, due to the gauge symmetries of the theory. Later we shall see that there are gauge conditions coupling the sectors by means of the definition of physical states. In this section, we deal with the so called local formulation, where the negative metric field contains physical degrees of freedom, and which is closer to the usual formulation. Later we will see that one can decouple the negative metric degrees of freedom in a way very similar to the one used in the Schwinger model case, but here the action will have further problems, which require the introduction of further degrees of freedom.

Local decoupled partition function and BRST symmetries.

We can obtain a factorized form of the partition function (3.52) by parametrizing $A_\pm$ as (3.40) as well as performing a chiral rotation,

$$\psi_1 \to \psi_1^{(0)} \equiv U\psi_1, \quad \psi_2 \to \psi_2^{(0)} = V^{-1}\psi_2 \quad . \tag{3.54}$$

Denoting the Jacobians by $\mathcal{J}_G[UV]$ and $\mathcal{J}_F[UV]$, and noting that the transformation decouples the fermions, that is,

$$\psi_1^\dagger(i\partial_+ + U^{-1}i\partial_+ U)\psi_1 = \psi_1^{(0)\dagger} i\partial_+ \psi_1^{(0)},$$
$$\psi_2^\dagger(i\partial_- + Vi\partial_- V^{-1})\psi_2 = \psi_2^{(0)\dagger} i\partial_- \psi_2^{(0)},$$

we arrive at the alternative form of the partition function (3.52),

$$Z = Z_F^{(0)} \int \mathcal{D}U\mathcal{D}W \mathcal{J}_G[W]\mathcal{J}_F[W] e^{i\mathcal{S}_{YM}[W]} \quad , \tag{3.55}$$

where $Z_F^{(0)}$ is the partition function of free fermions

$$Z_F^{(0)} = \int \mathcal{D}\psi^{(0)}\mathcal{D}\overline{\psi}^{(0)} \int \mathcal{D} e^{i\int d^2x \bar\psi i\partial\!\!\!/ \psi} \quad , \tag{3.56}$$

and $\mathcal{S}_{YM}$ is the Yang-Mills action

$$\mathcal{S}_{YM}[W] = -\frac{1}{4e^2}\int d^2x \,\mathrm{tr}\, \frac{1}{2}[\partial_+(Wi\partial_- W^{-1})]^2 = \int d^2x E^2 + E\partial_+(Wi\partial_- W^{-1}) \tag{3.57}$$
$$= -\frac{1}{4e^2}\int d^2x \,\mathrm{tr}\, \frac{1}{2}[\partial_-(W^{-1}i\partial_+ W)]^2 = \int d^2x E'^2 + E'\partial_-(W^{-1}i\partial_+ W) \quad , \tag{3.58}$$

with $W = UV$.

Under a vector gauge transformation U and V transform as $U \xrightarrow{G} UG$ and $V \xrightarrow{G} G^{-1}V$. Therefore, $W = UV$ is gauge invariant. In obtaining (3.57), (3.58) for the Yang Mills action we have first written the field strength tensor F_{01} in terms of U and V as

$$F_{01} = -\frac{1}{2}[D_+(U)Vi\partial_- V^{-1} - \partial_-(U^{-1}i\partial_+ U)] = \frac{1}{2}[D_-(V)U^{-1}i\partial_+ U - \partial_+(Vi\partial_- V^{-1})] \quad , \tag{3.59}$$

which can be rewritten in the two alternative forms

$$F_{01} = -\frac{1}{2}U^{-1}[\partial_+(Wi\partial_- W^{-1})]U = \frac{1}{2}V[\partial_-(W^{-1}i\partial_+ W)]V^{-1} \quad . \tag{3.60}$$

The logarithm of the Jacobian $\mathcal{J}_F$ is given, following Fujikawa, by

$$\ln \mathcal{J}_F = -\ln \frac{\det i D\!\!\!\!/}{\det i \ \partial\!\!\!/} = -i\Gamma[UV]. \tag{3.61}$$

We can prove the following assertion:

$$\mathcal{J}_G[UV] = e^{-ic_V \Gamma[UV]} (\det i\partial_+)_{adj} (\det i\partial_-)_{adj} \quad , \tag{3.62}$$

where c_V is the second Casimir of the group in question with the normalization $f_{acd}f_{bcd} = c_V\delta_{ab}$ of the structure constants. Representing $(\det i\partial_\pm)_{adj}$ in terms of ghosts and choosing the gauge $U = 1$, we obtain

$$\mathcal{Z} = \mathcal{Z}_F^{(0)} \mathcal{Z}_{gh}^{(0)} \mathcal{Z}_W \quad , \tag{3.63}$$

where $\mathcal{Z}_{gh}^{(0)} = \mathcal{Z}_{gh+}^{(0)}\mathcal{Z}_{gh-}^{(0)}$ and

$$\mathcal{Z}_W = \int \mathcal{D}W e^{-i(1+c_V)\Gamma[W]} e^{i\mathcal{S}_{YM}[W]} = \int \mathcal{D}W e^{-iS_{eff}[W]} \quad . \tag{3.64}$$

In the above expression the effective action is given by

$$S_{eff}[W] = S_{YM} - (c_V + 1)\Gamma[W] \tag{3.65}$$

and has two contributions, one corresponding to a WZW action with a negative coefficient, and the Yang-Mills action written in either of the forms (3.57) or (3.58).

We refer to (3.63) as the "local decoupled" partition function. As seen from (3.64), the ghosts $b_\pm^{(0)}$ are canonically conjugate to $c_\pm^{(0)}$ and have Grassmann parity +1. We assign to them the ghost number $gh\# = -1$ and $gh\# = +1$, respectively.

The dimensionality of the direct-product space $\mathcal{H}_F^{(0)} \otimes \mathcal{H}_{gh}^{(0)} \otimes \mathcal{H}_W$ associated with the partition function (3.63) is larger than that of the physical Hilbert space of the original fermionic formulation. Hence there must exist constrains imposing restrictions on the representations which are allowed in $\mathcal{H}_{phys}$. In order to discover these constraints we observe that the partition function is separately invariant under the following nilpotent transformations

$$\begin{array}{ll}
W\delta W^{-1} = -c_-^{(0)} \ , & W^{-1}\delta W = -c_+^{(0)} \ , \\
\delta\psi_1^{(0)} = c_-^{(0)}\psi_1^{(0)} \ , \ \delta\psi_2^{(0)} = 0 \ , & \delta\psi_1^{(0)} = 0 \ , \ \delta\psi_2^{(0)} = c_+^{(0)}\psi_2^{(0)} \ , \\
\delta c_-^{(0)} = \dfrac{1}{2}\{c_-^{(0)}, c_-^{(0)}\} \ , \ \delta c_+^{(0)} = 0 \ , & \delta c_-^{(0)} = 0 \ , \ \delta c_+^{(0)} = \dfrac{1}{2}\left\{c_+^{(0)}, c_+^{(0)}\right\} \ , \\
\delta b_-^{(0)} = \Omega_- \ , \ \delta b_+^{(0)} = 0 \quad , & \delta b_-^{(0)} = 0 \ , \ \delta b_+^{(0)} = \Omega_+ \ ,
\end{array} \tag{3.66}$$

where δ denotes the variation graded with respect to Grassmann parity, and $\Omega_\mp$ are given by

$$\begin{aligned}
\Omega_- &= -\frac{1}{4e^2}\mathcal{D}_-(W)(\partial_+(Wi\partial_- W^{-1})) - (1+c_V)\, J_-(W) + j_- \\
\Omega_+ &= -\frac{1}{4e^2}\mathcal{D}_+(W)(\partial_-(W^{-1}i\partial_+ W)) - (1+c_V)\, J_+(W) + j_+
\end{aligned} \tag{3.67}$$

with

$$J_-(W) = \frac{1}{4\pi}Wi\partial_- W^{-1} \quad , \quad J_+(W) = \frac{1}{4\pi}W^{-1}i\partial_+ W \tag{3.68}$$

$$j_- = \psi_1^{(0)}\psi_1^{(0)\dagger} + \{b_-^{(0)}, c_-^{(0)}\} \quad , \quad j_+ = \psi_2^{(0)}\psi_2^{(0)\dagger} + \{b_+^{(0)}, c_+^{(0)}\} \quad . \tag{3.69}$$

These transformations are easily derived by departing from two alternative forms (3.57) or (3.58) of the Yang-Mills action.

The corresponding BRST currents, as obtained via the usual Noether construction, are found to be

$$J_\mp = \mathrm{tr}\, c_-^{(0)} \left[\Omega_\mp - \frac{1}{2}\{b_\mp^{(0)}, c_\mp^{(0)}\}\right] \tag{3.70}$$

with $\partial_+ J_- = 0$, and $\partial_- J_+ = 0$.

Remarkably enough, the nilpotent symmetries lead to currents J_-, J_+ which only depend on the variable x^- and x^+, respectively.

The on-shell nilpotency of the corresponding conserved charges

$$Q_\pm = \int dx^1 J_\pm(x^\pm) \tag{3.71}$$

follows from the first-class character of the operators $\Omega^a_\pm = \mathrm{tr}\,(t^a \Omega_\pm)$.

3.4.2 QCD_2 in the non-local decoupled formulation and BRST constraints

The partition function represented by the standard expression (3.64) contains fields which are mixtures of massive and massless modes, of positive and negative norms respectively, coupled by the constraints. In the following section we dissociate these degrees of freedom by a suitable transformation. We shall thereby be lead to an alternative nonlocal representation of the partition function, useful for learning certain structural properties.

Non-local decoupled partition function and BRST symmetries

Following ref. [42], we make in (3.57) the change of variable $E \to \beta$ defined by

$$\partial_+ E = \left(\frac{1+c_V}{2\pi}\right) \beta^{-1} i\partial_+ \beta \quad . \tag{3.72}$$

The Jacobian associated with this change of variables is

$$\mathcal{D}E = \det i\mathcal{D}_+(\beta)\mathcal{D}\beta \tag{3.73}$$

where we have suppressed the constant $\det \partial_+$ which will not play any role in the discussion to follow. Making use of the determinant of the fermionic operator in the adjoint representation (which corresponds to the determinant in the fundamental representation raised to the power c_V) as well as (3.61), and representing $(\det i\partial_+)$ as a functional integral over ghost fields $\hat{b}_-$ and $\hat{c}_-$, we have, after decoupling the ghosts,

$$\begin{aligned} Z =& Z_F^{(0)} Z_{gh}^{(0)} \hat{Z}_{gh-}^{(0)} \int \mathcal{D}W \int \mathcal{D}\beta \exp\{-i(1+c_V)[\Gamma[W] + \Gamma[\beta] \\ & -\frac{1}{4\pi}\int tr(\beta^{-1}\partial_+\beta W \partial_- W^{-1})]\} \\ & \times \exp(i\Gamma[\beta]) \exp\left\{ i\left(\frac{1+c_V}{2\pi}\right)^2 e^2 \int \frac{1}{2} tr\left[\partial_+^{-1}(\beta^{-1}\partial_+\beta\right]^2 \right\} \end{aligned} \tag{3.74}$$

where

$$\hat{Z}_{gh-}^{(0)} = \int \mathcal{D}\hat{b}_-^{(0)} \mathcal{D}\hat{c}_-^{(0)} e^{i\int d^2x\, \mathrm{tr}\, \hat{b}_-^{(0)} i\partial_+ \hat{c}_-^{(0)}} \tag{3.75}$$

Using the Polyakov-Wiegmann identity (3.22) and making the change of variable $W \to \beta W = \tilde{W}$, we are left with

$$Z = Z_F^{(0)} Z_{gh}^{(0)} \hat{Z}_{gh-}^{(0)} Z_{\tilde{W}} Z_\beta \tag{3.76}$$

where

$$Z_\beta = \int \mathcal{D}\beta \exp\left\{ i\Gamma[\beta] + i\left(\frac{1+c_V}{2\pi}\right)^2 e^2 \int \frac{1}{2} tr\left[\partial_+^{-1}(\beta^{-1}\partial_+\beta)\right]^2 \right\} \tag{3.77}$$

and

$$Z_{\tilde{W}} = \int \mathcal{D}\tilde{W} \exp[-i(1+c_V)\Gamma[\tilde{W}]] \tag{3.78}$$

is the partition function of a WZW field of level $-(1+c_V)$. Note that the decoupling of the β-field depended crucially on the choice of the multiplicative constant in (3.72). Repeating this process by starting from expression (3.58) and making now the change of variables

$$\partial_+ E' = \left(\frac{1+c_V}{2\pi}\right) \beta' i\partial_- \beta'^{-1} \tag{3.79}$$

and

$$W \to W\beta' = \tilde{W}' \quad . \tag{3.80}$$

one arrives at the equivalent representation

$$Z = Z_F^{(0)} Z_{gh}^{(0)} \hat{Z}_{gh+}^{(0)} Z_{\tilde{W}'} Z_{\beta'} \tag{3.81}$$

where

$$\hat{Z}_{gh+}^{(0)} = \int \mathcal{D}\hat{b}_+^{(0)} \mathcal{D}\hat{c}_+^{(0)} e^{i\int d^2x \, \mathrm{tr}\, \hat{b}_+^{(0)} i\partial_- \hat{c}_+^{(0)}} \quad , \tag{3.82}$$

$$Z_{\beta'} = \int \mathcal{D}\beta' \exp\left\{ i\Gamma[\beta'] + i\left(\frac{1+c_V}{2\pi}\right)^2 e^2 \int \frac{1}{2} tr\left[\partial_+^{-1}(\beta'\partial_+\beta'^{-1})\right]^2 \right\} \quad , \tag{3.83}$$

$$Z_{\tilde{W}'} = \int \mathcal{D}\tilde{W}' \exp[-i(1+c_V)\Gamma[\tilde{V}]] \quad . \tag{3.84}$$

The partition function (3.81) exhibits nilpotent symmetries in a variety of sectors, not all of which are to be imposed as symmetries of the physical states. In the following section we systematically derive those nilpotent symmetries which have to be realized in order that the theory is consistently defined and represents two-dimensional QCD.

Systematic derivation of the constraints

The main objective of this section is to trace the fate of the BRST conditions of the local formulation, when going over to the so-called[42] non-local formulation. We shall also establish from first principles further BRST conditions that may have to be imposed in order to ensure the equivalence of local and non-local formulations.

a) The BRST condition $Q_+ \approx 0$ in the non-local formulation

Making use of some algebraic identities, we may rewrite Ω_+ in (3.67) as

$$\Omega_+ = -\frac{1}{4e^2} W^{-1}\left[\partial_+^2 (W i\partial_- W^{-1})\right] W - (1+c_V)J_+(W) + j_+. \tag{3.85}$$

Using the equation of motion for E, Ω_+ and making the change of variable (3.72), we then obtain

$$\Omega_+ = -(1+c_V)J_+(\tilde{W}) + j_+, \tag{3.86}$$

where $\tilde{W} = \beta W$. We conclude that the corresponding nilpotent charge

$$Q_+ = \int dx^1 tr c_+^{(0)} \left[-(1+c_V)J_+(\tilde{W}) + j_+ - \frac{1}{2}\left\{b_+^{(0)}, c_+^{(0)}\right\}\right], \tag{3.87}$$

must annihilate the physical states.

b) The BRST condition $Q_- \approx 0$ in the non-local formulation

In the case of the BRST charge Q_+, the symmetry transformations in the V-fermion-ghost space giving rise to this conserved charge could be trivially extended to the $E-W$-fermion-ghost space. This is not true for Q_-, where the BRST symmetry for E is maintained off-shell only by adding an extra (commutator) term (which vanishes for E "on shell"). One is thereby led to a fairly complicated expression for Q_- in terms of the variables $\beta, \tilde{W}$ of the non-local formulation.

A more transparent result is obtained by rewriting Ω_- in terms of the variables $\tilde{W}'$ defined in (3.80). Making use of the E' equation of motion and making the corresponding change of variables (3.79), we may rewrite Ω_- in (3.67) in the form

$$\Omega_- = -(1+c_V)J_-(\tilde{W}') + j_-. \tag{3.88}$$

We conclude that the corresponding BRST charge

$$Q_- = \int dx^1 tr c_-^{(0)}[-(1+c_V)J_-(\tilde{W}') + j_- \frac{1}{2}\{b_-^{(0)}, c_-^{(0)}\}] \tag{3.89}$$

must annihilate the physical states.

Although $\tilde{W}$ in (3.87) and $\tilde{W}'$ in (3.89) obey the same dynamics, as described by the WZW action, they are related by different constraints to the "massive" sector, which is described in terms of the group-valued fields β and β', respectively, obeying different dynamics. We will return to this question in the next section.

c) BRST condition associated with the changes of variables $E \to \beta$ and $E' \to \beta'$.

The change of variable (3.72) leads to a BRST condition on the physical states. We follow the procedure outlined in ref. [43], arriving at the constraint $\tilde{\Omega}_-$, given by

$$\begin{aligned}\hat{\Omega}_- =& -\lambda^2 \beta(\partial_+^{-2}(\beta^{-1} i\partial_+ \beta))\beta^{-1} + J_-(\beta) \\ & -(1+c_V)J_-(\tilde{W}) + \left\{\hat{b}_-^{(0)}, \hat{c}_-^{(0)}\right\}.\end{aligned} \tag{3.90}$$

The corresponding Noether current is found to be

$$\hat{J}_- = \mathrm{tr}\,\hat{c}_-^{(0)}\left[\hat{\Omega}_- - \frac{1}{2}\left\{b_-^{(0)}\hat{c}_-^{(0)}\right\}\right]. \tag{3.91}$$

with $\partial_+\hat{J}_- = 0$. Our deductive procedure shows that the corresponding nilpotent charge $\hat{Q}_-$ must annihilate the physical states:

$$\hat{Q}_- = 0 \quad \text{on} \quad \mathcal{H}_{phys}. \tag{3.92}$$

Repeating this procedure for the change of variable (3.79) in (3.58), one is led to conclude that the left-moving current

$$\hat{J}_+ = \mathrm{tr}\,\hat{c}_+^{(0)}\left[\hat{\Omega}_+ - \frac{1}{2}\left\{b_+^{(0)}\hat{c}_+^{(0)}\right\}\right] \quad , \tag{3.93}$$

which obeys $\partial_-\hat{J}_+ = 0$, and where

$$\begin{aligned}\hat{\Omega}_+ =& -\lambda^2\beta'^{-1}(\partial_-^{-2}(\beta' i\partial_-\beta'^{-1}))\beta' + J_+(\beta') \\ & -(1+c_V)J_+(\tilde{V}') + \{\hat{b}_+^{(0)}, \hat{c}_+^{(0)}\}.\end{aligned} \tag{3.94}$$

must also annihilate the physical states. On the zero-ghost-number sector conditions $\hat{Q}_\pm \approx 0$ are again equivalent to requiring $\hat{\Omega}_\pm \approx 0$. In summary, on the zero-ghost-number sector, the BRST conditions which should be imposed on the physical states are equivalent to requiring

$$\Omega_\pm \approx 0, \quad \hat{\Omega}_\pm \approx 0, \tag{3.95}$$

with $\hat{\Omega}_\pm$ and $\Omega_\pm$ given by eqs. (3.86), (3.88), (3.90), and (3.94). It is interesting to observe that the gauging procedure of [44] would suggest the existence of further constraints. It thus merely provides a guideline for discovering candidates to constraints.

3.4.3 The physical Hilbert space

In order to address the cohomology problem defining the physical Hilbert space, we must express the constraints in terms of canonically conjugate variables. We first rewrite the partition function Z_β in (3.77) in terms of an auxiliary field B as

$$Z_\beta = \int \mathcal{D}B\mathcal{D}\beta e^{iS[\beta,B]}, \tag{3.96}$$

where

$$S[\beta, B] = \Gamma[\beta] + \int tr\left[\frac{1}{2}(\partial_+B)^2 + \lambda B\beta^{-1}i\partial_+\beta\right]. \tag{3.97}$$

Similarly, we have for $Z_{\beta'}$ in (3.83)

$$Z_{\beta'} = \int \mathcal{D}B'\mathcal{D}\beta' e^{iS'[\beta',B']}, \tag{3.98}$$

with

$$S'[\beta', B'] = \Gamma[\beta'] + \int tr\left[\frac{1}{2}(\partial_- B')^2 + \lambda B'\beta' i\partial_-\beta'^{-1}\right]. \tag{3.99}$$

We may then rewrite the constraints $\hat{\Omega}_\pm \approx 0$ in (3.90) and (3.94) as

$$\hat{\Omega}_- = \lambda\beta B\beta^{-1} + \frac{1}{4\pi}\beta i\partial_-\beta^{-1} - \frac{(1+c_V)}{4\pi}\tilde{V} i\partial_-\tilde{V}^{-1} + \{\hat{b}_-^{(0)}, \hat{c}_-^{(0)}\}, \tag{3.100}$$

$$\hat{\Omega}_+ = \lambda\beta'^{-1}B'\beta' + \frac{1}{4\pi}\beta'^{-1} i\partial_+\beta' - \frac{(1+c_V)}{4\pi}\tilde{V'}^{-1} i\partial_+\tilde{V}' + \{\hat{b}_+^{(0)}, \hat{c}_+^{(0)}\}. \tag{3.101}$$

Canonical quantization then allows one to obtain the Poisson algebra. It is straightforward to verify that $\hat{\Omega}^a_+ = tr(\hat{\Omega}t^a)$ and $\hat{\Omega}^a_- = tr(\hat{\Omega}t^a)$ are first class:

$$\left\{\hat{\Omega}^a_\pm(x), \hat{\Omega}^b_\pm(y)\right\}_{PB} = -f_{abc}\hat{\Omega}^c_\pm\delta(x^1 - y^1). \tag{3.102}$$

Hence the corresponding BRST charges are nilpotent. Similar properties are readily established for the remaining operators $\Omega_\pm$. Furthermore,

$$\{\Omega_+(x), \hat{\Omega}_-(y)\}_{PB} = 0 \quad , \quad \{\Omega_-(x), \hat{\Omega}_+(y)\}_{PB} = 0 \quad . \tag{3.103}$$

The physical Hilbert space of the non-local formulation of QCD_2 is now obtained by solving the cohomology problem associated with the BRST charges $Q_\pm, \hat{Q}_\pm$ in the zero-ghost-number sector. The solution of this problem is suggested by identifying this space with the space of gauge-invariant observables of the original theory. It is interesting to note that the constraints $\hat{\Omega}_\pm \approx 0$ are implemented by any functional of V (and the fermions), thus implying that $\tilde{V}, \beta(\tilde{V}', \beta')$ can only occur in the combinations $\beta^{-1}\tilde{V}(\tilde{V}'\beta'^{-1})$. Indeed, making use of the Poisson brackets defined by the canonical procedure, we have

$$\begin{aligned}
\left\{\hat{\Omega}^a_-(x),\ \beta^{-1}(y)\right\}_{PB} &= i(\beta^{-1}(x)t^a)\delta(x^1 - y^1),\\
\left\{\hat{\Omega}^a_-(x),\ \tilde{V}(y)\right\}_{PB} &= -i(t^a\tilde{V}(y))\delta(x^1 - y^1),\\
\left\{\hat{\Omega}^a_+(x),\ \tilde{V}'(y)\right\}_{PB} &= +i(\tilde{V}'(x)t^a)\delta(x^1 - y^1),\\
\left\{\hat{\Omega}^a_+(x),\ \beta'^{-1}(y)\right\}_{PB} &= -i(t^a\beta'^{-1}(y))\delta(x^1 - y^1) \quad .
\end{aligned} \tag{3.104}$$

As for the other two constraints, $\Omega_+ \approx 0$ and $\Omega_- \approx 0$, which link the bosonic to the free fermionic sector, they indicate that local fermionic bilinears should be constructed in terms of free fermions and the bosonic fields as

$$\left(\psi_1^{(0)\dagger}\beta^{-1}\tilde{V}\psi_2^{(0)}\right) = \left(\psi_1^{(0)\dagger}\tilde{V}'\beta'^{-1}\psi_2^{(0)}\right) = \left(\psi_1^{(0)\dagger}V\psi_2^{(0)}\right) = \left(\psi_1^\dagger\psi_2\right). \tag{3.105}$$

This is in agreement with our expectations.

3.5 Massive two-dimensional QCD

In this section we shall see that the BRST symmetries of the physical states in massless QCD_2 are also the symmetries which should be imposed on the physical states in the massive case.

For massive fermions the functional determinant of the Dirac operator, an essential ingredient for arriving at the bosonised form of the QCD_2 partition function, can no longer be computed in closed form, and one must resort to the so-called adiabatic principle of form invariance. Equivalently, one can start with a perturbative expansion in powers of the mass, as given by

$$\sum \frac{1}{n!} M^n \left[\int d^2x \overline{\psi}\psi\right]^n \quad , \tag{3.106}$$

use the (massless) bosonization formulae and re-exponentiate the result. In this approach, the mass term is given in terms of a bosonic field g_ψ of the massless theory by[45]

$$S_m = -M \int \overline{\psi}\psi = M\mu \int \mathrm{tr}(g_\psi + g_\psi^{-1}) \quad ,$$

where μ is an arbitrary massive parameter whose value depends on the renormalization prescription for the mass operator.[1]

Defining $m^2 = M\mu$, we re-exponentiate the mass term. Going through the changes of variables leading to (3.76) and (3.81), one arrives at the following alternative forms for the mass term when expressed in terms of the fields of the non-local formulation

$$S_m = m^2 \int \mathrm{tr}(g\tilde{\Sigma}^{-1}\beta + \beta^{-1}\tilde{\Sigma}g^{-1}) \quad , \tag{3.107}$$

$$S'_m = m^2 \int \mathrm{tr}(g\beta'\tilde{\Sigma}'^{-1} + \tilde{\Sigma}'\beta'^{-1}g^{-1}) \quad . \tag{3.108}$$

The corresponding effective action of massive QCD_2 in the non-local formulation reads

$$S = S_{YM}[\beta, B] + S_m[g, \beta, \tilde{\Sigma}] + \Gamma[g] + \Gamma[\beta] - (c_V + 1)\Gamma[\tilde{\Sigma}] + S_{gh} + \hat{S}_{gh-} \quad , \tag{3.109}$$

$$S' = S'_{YM}[\beta', B'] + S_m[g, \beta', \tilde{\Sigma}'] + \Gamma[g] + \Gamma[\beta'] - (c_V + 1)\Gamma[\tilde{\Sigma}'] + S_{gh} + \hat{S}_{gh+} \quad . \tag{3.110}$$

We thus see that the associated partition function no longer factorizes. Nevertheless, there still exist BRST currents which are either right- or left-moving, just as in the massless case.

The actions (3.109) and (3.110) exhibit various symmetries of the BRST type; however, not all of them lead to nilpotent charges. The variations are graded with respect to Grassmann number. The equations of motion obtained from action (3.109) read

$$\frac{1}{4\pi}\partial_+(g\partial_- g^{-1}) = m^2(g\tilde{\Sigma}^{-1}\beta - \beta^{-1}\tilde{\Sigma}g^{-1}) \quad , \tag{3.111}$$

$$-\frac{c_V + 1}{4\pi}\partial_+(\tilde{\Sigma}\partial_-\tilde{\Sigma}^{-1}) = m^2(\tilde{\Sigma}g^{-1}\beta^{-1} - \beta g\tilde{\Sigma}^{-1}) \quad , \tag{3.112}$$

$$\frac{1}{4\pi}\partial_+(\beta\partial_-\beta^{-1}) + i\lambda\partial_+(\beta B\beta^{-1}) = m^2(\beta g\tilde{\Sigma}^{-1} - \tilde{\Sigma}g^{-1}\beta^{-1}) \quad , \tag{3.113}$$

$$-\frac{1}{4\pi}\partial_-(\beta^{-1}\partial_+\beta) + i\lambda[\beta^{-1}\partial_+\beta, B] + i\lambda\partial_+ B = m^2(g\tilde{\Sigma}^{-1}\beta - \beta^{-1}\tilde{\Sigma}g^{-1}) \quad , \tag{3.114}$$

$$\partial_+^2 B = \lambda(\beta^{-1}i\partial_+\beta) \quad , \tag{3.115}$$

$$\partial_\pm b_\mp = 0 \quad , \quad \partial_\pm c_\mp = 0 \quad , \tag{3.116}$$

with an analogous set of equations involving the prime sector. Notice that the mass term can be transformed from one equation to another, by a suitable conjugation. Making use of eqs. (3.111-3.116), the Noether currents are constructed in the standard fashion: we make a general BRST variation of the action, without using the equations of motion, and equate the result to the on-shell variation, taking into account terms arising from partial integrations. The only subtlety in this procedure concerns the WZW term, which only contributes off-shell to the variation. The four conserved Noether currents are found to be

$$J_\pm = \mathrm{tr}\left(c_\pm\Omega_+ - \frac{1}{2}b_\pm\{c_\pm, c_\pm\}\right) \quad , \tag{3.117}$$

$$\hat{J}_\pm = \mathrm{tr}\left(c_\pm\hat{\Omega}_\pm - \frac{1}{2}\hat{b}_\pm\{\hat{c}_\pm, \hat{c}_\pm\}\right) \quad , \tag{3.118}$$

where the Ω's are given by

$$\Omega_+ = \left(\frac{1}{4\pi}g^{-1}i\partial_+ g - \frac{c_V+1}{4\pi}\tilde{\Sigma}^{-1}i\partial_+\tilde{\Sigma} + \{b_+, c_+\}\right) \quad , \tag{3.119}$$

$$\Omega_- = \left(\frac{1}{4\pi}gi\partial_- g^{-1} - \frac{c_V+1}{4\pi}\tilde{\Sigma}' i\partial_-\tilde{\Sigma}'^{-1} + \{b_-, c_-\}\right) \quad , \tag{3.120}$$

$$\hat{\Omega}_- = \left(\frac{1}{4\pi}\beta i\partial_-\beta^{-1} - \frac{c_V+1}{4\pi}\tilde{\Sigma}i\partial_-\tilde{\Sigma}^{-1} - \lambda\beta B\beta^{-1} + \{b_-, c_-\}\right) \quad , \tag{3.121}$$

$$\hat{\Omega}_+ = \left(\frac{1}{4\pi}\beta'^{-1}i\partial_+\beta' - \frac{c_V+1}{4\pi}\tilde{\Sigma}'^{-1}i\partial_+\tilde{\Sigma}' - \lambda\beta'^{-1}B'\beta' + \{\hat{b}_+, \hat{c}_+\}\right) \quad . \tag{3.122}$$

From the current conservation laws

$$\partial_\mp J_\pm = 0 \quad , \quad \partial_\pm \hat{J}_\mp = 0 \quad , \tag{3.123}$$

one infers that Ω_- and $\hat{\Omega}_-$ are right-moving, while Ω_+ and $\hat{\Omega}_+$ are left-moving. Indeed, making use of the equations of motion (3.111-3.116) one readily checks that the operators $\Omega_\pm$, $\hat{\Omega}_\pm$ satisfy

$$\partial_\mp\Omega_\pm = 0 \quad , \quad \partial_\pm\hat{\Omega}\mp = 0 \quad , \tag{3.124}$$

consistent with the conservation laws (3.123).

3.6 Screening in two-dimensional QCD

In this section we reconsider the problem of screening and confinement. We shall concentrate on the case of single flavour QCD, and merely comment on the general case at the end of the section.

We proceed by first considering the case of massless fermions and compute the inter-quark potential. We introduce a pair of classical colour charges of strength $q = q^a t^a$ separated by a distance L. Such a pair is introduced in the action (3.109) by means of the substitution

$$i(\beta^{-1}\partial_+\beta)^a \longrightarrow i(\beta^{-1}\partial_+\beta)^a - \frac{2\pi}{e} q^a\Big(\delta(x-\frac{L}{2}) - \delta(x+\frac{L}{2})\Big) \quad , \tag{3.125}$$

where a is a definite colour index. This adds the following term to the action [1]

$$V(L) = \Delta S = S_q - S = -(c_V+1)q^a\Big(B^a(L/2) - B^a(-L/2)\Big) \quad . \tag{3.126}$$

The equation of motion for B^a is now replaced by

$$\partial_+^2 B^a = i\lambda(\beta^{-1}\partial_+\beta)^a - (c_V+1)q^a\Big(\delta(x-\frac{L}{2}) - \delta(x+\frac{L}{2})\Big), \tag{3.127}$$

which implies, upon substitution into the equation of motion for the β-field,

$$\begin{aligned} \partial_+\left(\frac{i}{4\pi\lambda}\partial_-\partial_+B + [\partial_+B, B] + i\lambda B\right) &= \\ \left(\frac{-iq}{2e}\partial_- + (c_V+1)[q,B]\right)[\delta(x-\frac{L}{2}) - \delta(x+\frac{L}{2})] \quad . \end{aligned} \tag{3.128}$$

We look for solutions of (3.128) with a fixed global orientation in colour space.[2] We thus make $B^a = q^a f(x)$. This renders the problem abelian. We thus infer that the potential (3.126) has the form

$$V(L) = \frac{(c_v+1)\sqrt{\pi}}{2}\frac{q^2}{e}(1 - e^{-2\sqrt{\pi}\lambda L}) \tag{3.129}$$

which implies that the system is in a screening phase.

We now turn to the case of massive fermions. Taking the external charge to lie in the direction t^2 of $SU(n)$ space, our Ansatz for B^a leads one to look for solutions with g, β and Σ parametrized as

$$g = e^{i2\sqrt{\pi}\varphi\sigma_2}, \qquad \beta = e^{i2\sqrt{\pi}E\sigma_2}, \qquad \Sigma = e^{-i2\sqrt{\pi}\eta\sigma_2} \quad , \tag{3.130}$$

The equations of motion (3.111-3.115) are replaced by[3]

$$\partial_+\partial_-\varphi = -4\sqrt{\pi}m^2 \sin 2\sqrt{\pi}(E+\varphi+\eta) \quad , \tag{3.131}$$

$$\partial_+\partial_-\eta = \frac{4\sqrt{\pi}}{c_v+1}m^2 \sin 2\sqrt{\pi}(E+\varphi+\eta), \tag{3.132}$$

$$\begin{aligned} \partial_+\partial_-E + 4\pi\lambda^2 E &= -4\sqrt{\pi}m^2 \sin 2\sqrt{\pi}(E+\varphi+\eta) \\ &\quad -2\sqrt{\pi}(c_v+1)\lambda q\Big[\Theta(x+\frac{L}{2}) - \Theta(x-\frac{L}{2})\Big] \quad . \end{aligned} \tag{3.133}$$

[1] This corresponds to minus the same term added to the hamiltonian.

[2] Note that this is a non-trivial input, since we have no longer the freedom of choosing a gauge in which such an Ansatz could be realized

[3] We leave the Casimir c_V as a free parameter, since the expressions corresponding to the Schwinger model will simply be obtained from the SU(N) model by taking the limit $c_V \to 0$.

We notice the combination $\psi = \varphi + (c_v + 1)\eta$ describes a zero-mass field. In order to compute the potential we consider the static limit. This is equivalent to considering the effective lagrangean

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}\partial_1 E^2 + \frac{1}{2}\frac{c_V+1}{c_V}\partial_1\Phi^2 + 2\pi\lambda^2 E^2 - 2m^2 \cos 2\sqrt{\pi}(\Phi+E)\\ &+2\sqrt{\pi}(c_V+1)\lambda q E\left[\Theta(x+\frac{L}{2}) - \Theta(x-\frac{L}{2})\right]\\ &-\frac{(c_V+1)^2 q^2}{2}\left[\Theta(x+\frac{L}{2}) - \Theta(x-\frac{L}{2})\right] - \frac{\partial_1\psi^2}{2c_v} \quad ,\end{aligned} \tag{3.134}$$

where $\Phi = \varphi + \eta$. In order to compute the inter-quark potential, we shall expand the cosine in the effective Lagrangian (3.134) up to second order in the argument. This pre-supposes a bound in the fluctuations of the fields. We can confirm that the solution is consistent with such a condition.

In the weak-limit approximation, we expand the cosine term. Consequently, we diagonalise the hamiltonian and solve the equations of motion.[4] The diagonalisation of the quadratic lagrangean leads to the expression

$$\begin{aligned}\mathcal{L}=&-\frac{1}{2c_V}\partial_1\psi^2 + (1+\epsilon a^2)\left\{\frac{1}{2}\chi'^2_+ + \frac{1}{2}m_+^2\chi_+^2 + \lambda Q_+\chi_+\right\}\\ &+\frac{(1+\epsilon a^2)}{a^2}\left\{\frac{1}{2}\partial_1\chi_-^2 + \frac{1}{2}m_-^2\chi_-^2 + \lambda Q_-\chi_-\right\} \quad ,\end{aligned} \tag{3.135}$$

where we have found it useful to define the following variables :

$$\chi_+ = \frac{1}{1+\epsilon a^2}(E - a\Phi) \quad , \quad \chi_- = \frac{1}{1+\epsilon a^2}(\Phi + \epsilon a E) \tag{3.136}$$

and the parameters:

$$\begin{aligned}\epsilon=&\frac{c_V}{(c_v+1)} \quad , \quad q_+ = \frac{2\sqrt{\pi}(c_v+1)q}{(1+\epsilon a^2)},\\ a=&-\frac{8\pi m^2}{m_+^2 - 16\epsilon m^2} \quad , \quad q_- = \frac{2\sqrt{\pi}\epsilon a(c_v+1)q}{(1+\epsilon a^2)}\\ m_\pm^2=&2\pi[\left(\lambda^2 + (1+\epsilon)2m^2\right) \pm \sqrt{(\lambda^2+(1+\epsilon)2m^2)^2 - 8\epsilon\lambda^2 m^2}],\\ Q_\pm=&q_\pm\left[\Theta(x-\frac{L}{2}) - \Theta(x+\frac{L}{2})\right] \quad .\end{aligned} \tag{3.137}$$

Solving the corresponding equations of motion yields:

$$\chi_\pm = \frac{\lambda q_\pm}{m_\pm^2}\begin{cases}\sinh(m_\pm \frac{L}{2})e^{-m_\pm|x|} & |x| > \frac{L}{2}\\ (1 - e^{-m_\pm L/2}\cosh m_\pm x) & |x| < \frac{L}{2}\end{cases} \tag{3.138}$$

[4]All the forthcoming computations will in general be valid for any compact group. In such cases, the mass term can always be expanded in terms of algebra-valued fields after a convenient parametrisation.

from which we obtain the inter-quark potential energy

$$\begin{aligned} V(L) =& \frac{(c_V+1)^2 q^2}{2} \\ &\times \left[\left(\frac{4\pi\lambda^2 - m_-^2}{m_+^2 - m_-^2} \right) \left(\frac{1 - e^{-m_+ L}}{m_+} \right) + \left(\frac{m_+^2 - 4\pi\lambda^2}{m_+^2 - m_-^2} \right) \left(\frac{1 - e^{-m_- L}}{m_-} \right) \right] . \end{aligned} \tag{3.139}$$

Thus we find two mass scales given by m_+ and m_-. Both these scales correspond to screening-type contributions if $c_V \neq 0$.

Next, we compare the results with those obtained for the Schwinger model. In the abelian case, the combination of the matter boson φ and the negative metric scalar η gives rise to the θ-angle. That is, the combination $\Phi \equiv \varphi + \eta = \theta$ appears in the mass term. When fermions are massless, the electric field and the matter boson decouple. However, due to a Higgs mechanism, the electric field acquires a mass and, therefore, a long-range force does not exist. This leads to a pure screening potential. On the other hand, for massive fermions, the electric field couples to the matter boson Φ. Yet, $\Phi = \psi_{c_V=0}$, and hence, it remains massless. This coupling to Φ via the mass term is the origin of the long-range force (linearly rising potential) in the massive U(1) case. Therefore the potential is confining.

On the other hand, the expression (3.139) for the potential indicates the absence of a long-range force in the non-abelian case. This can be understood by recalling that in such a case $\Phi \neq \psi$, so that Φ describes a massive field. The massless field ψ decouples from the electric field (see eqn. (3.133)). The massive field $\Phi \equiv \varphi + \eta$, is the combination that couples to E. Therefore, as both E and Φ are massive, there is no long-range force. This is confirmed by our explicit computations.

The abelian potential can also be obtained from (3.139) by taking the limit $c_V \to 0$. In this limit, the mass scale m_- tends to zero and we recover the linearly rising potential, signaling confinement.

It is interesting to examine the behaviour of the screening potential (3.139) in extreme limits. In the strong coupling regime, $\lambda^2 >> m^2$, the mass parameter m_+ dominates ($m_+ >> m_-$) and we have

$$V(L)_{(m<<e)} \simeq \frac{(c_V+1)^2 q^2}{2} \left\{ \frac{(1 - e^{-2\sqrt{\pi}\lambda L})}{2\sqrt{\pi}\lambda} + \frac{\sqrt{\pi\epsilon} m}{\lambda^2} (1 - e^{-2\sqrt{2\pi\epsilon} m L}) \right\} . \tag{3.140}$$

On the other hand, in the weak coupling limit, $m >> e$, we obtain

$$V(L)_{(m>>e)} \simeq \frac{(c_V+1) q^2}{4\sqrt{\pi}\lambda} \sqrt{\frac{1+\epsilon}{\epsilon}} \left(1 - e^{-2\sqrt{\pi\epsilon/(1+\epsilon)}\lambda L} \right) . \tag{3.141}$$

In both regimes, the potential is governed by the parameter λ, *i.e.* by the coupling constant.

Adding flavour

So far, we have considered fermions in the fundamental representation of $U(N)$. Explicit bosonisation formulas for fields in a higher representations are in general not available. We

can however proceed by introducing F copies $\{\psi^i_f\} = \psi^i_1\psi^i_2, \cdots \psi^i_F$ of the $U(N)$ fermionic fields, labelled by a flavour quantum number f. We then treat the mass term perturbatively, introducing it via the Coleman's principle of form-invariance. The corresponding effective action is a simple generalization of (3.109)), with a set of F matrix valued fields $g_f, f = 1, \cdots F$, each in the fundamental representation of $SU(N)_L \times SU(N)_R$.

Following the procedure of the last section, we fix the external charges in colour space, parametrise the fields as in equation (36) and take the weak-field and static limit, to arrive at the same conclusions as before, namely, at the screening phase.

Exotic states

In the preceding sections, we have performed a semi-classical analysis in order to understand the mechanism of screening and confinement in two-dimensional QCD. In order to distinguish between the different phases, we have used a dipole-dissociation test. If the particles are confined, an infinite amount of energy is required to isolate them. In this case, as the inter-quark distance increases, pair production occurs which obscures the physical interpretation of the results. On the other hand, in the screening phase the amount of energy required to dissociate the dipole is finite. Although charge (or colour) cannot be seen because of vacuum polarisation, further structures (or quantum numbers) can be observed.

In this section, we outline the construction of eigenstates of the hamiltonian which carry flavour quantum numbers. These are the analogues of the exotic states in the Schwinger model[7]. Such a discussion provides a more elaborate confirmation of the screening phase.

We construct the exotic states by means of the fermionic operator [1].

$$\mathcal{F}_f(x) = \prod_a e^{i\sqrt{\pi}\phi^a_f(x^0,x^1)/(c_v+k) + i(c_v+k)\sqrt{\pi}\int_{-\infty}^{x^1} \dot{\phi}^a_f(x^0,y^1)dy^1} = \prod_a \mathcal{F}^a_f, \tag{3.142}$$

where the field ϕ^a_f does not carry colour charge.[5] From the semi-classical discussion, the combination $(\varphi^a_f + \eta)$ is the natural candidate for the operator ϕ^a_f. This is because we have chosen symmetric boundary conditions $(\phi^a_f(+\infty) = \phi^a(-\infty))$ which imply that ϕ^a_f carries no charge. In the quantum theory[46], the operator $\phi^a_f = \varphi^a_f + \eta$ appears in the BRST current

$$J_+ = c_+ \left(ig_f^{-1}\partial_+ g_f - i(c_v + k)\Sigma^{-1}\partial_+\Sigma + \text{ghosts} \right), \tag{3.143}$$

which is conserved (actually vanishing) and leads to the topological charges

$$Q = \left(\sum_{f=1}^{k} \varphi_f + (c_v + k)\eta \right)(t, \infty) - \left(\sum_{f=1}^{k} \varphi_f + (c_v + k)\eta \right)(t, -\infty) + \cdots . \tag{3.144}$$

where the dots stand for commutator-type corrections.

[5] We do not expect (3.142) to be the complete operator which describes flavoured physical states. Corrections involving multiple commutators, due to the non-abelian character of the theory, can appear.

Next, we argue that the operator (3.142) commutes with the mass term. We use the parametrisation

$$g = e^{i\varphi^1\sigma^1} e^{i\varphi^2\sigma^2} e^{i\varphi^3\sigma^3}, \tag{3.145}$$

$$\Sigma = e^{-i\eta^1\sigma^1} e^{-i\eta^2\sigma^2} e^{-i\eta^3\sigma^3}, \tag{3.146}$$

in the SU(N) model and take the commutator of $\mathcal{F}_f$ with the mass term. This shifts φ^a by $2\pi(c_v + k)$, and η^a by 2π. Since SU(2) is a compact group, we conclude that $\mathcal{F}_f$ commutes with the hamiltonian. This result can be generalized to any SU(N) gauge group.

By comparing expression (3.142) with the fermionic operator, we see that the field η plays a rôle similar to that played previously by the sum $\sum_{i=1}^{k} \psi_k$ in the abelian theory. Consequently, the fields are not constrained in the non-abelian model and enjoy canonical commutation relations. Thus, kink dressing might be needed [47, 48]. In addition, the θ-vacuum does not enter the expression for the fermionic operator (3.142) in the non-abelian theory.

Validity of the semi-classical approach and prospects about the four-dimensional theory

The discussion of previous sections is based on a semi-classical approach to QCD_2, which in view of the subtleties linked to the quantum behaviour of the theory, may lead to doubts about the validity of the method, especially concerning application to the four dimensional case. However we shall argue that we get the correct picture in the two-dimensional case and that the problem can be pursued in four dimensions.

First some brief prolegomena, which concern the bosonisation procedure. It is well known that the bosonised theory contains quantum information at the classical level. Indeed, the anomaly equation, which is a one-loop effect in the fermionic theory, is contained in the classical field equation of the bosonic field. We also add the fact that most interesting effects concerning two-dimensional gauge theories are one loop effects, such as mass generation for the gauge field (coming directly from the anomaly equation, thus a classical effect in the bosonic language) or the vacuum structure, which in the bosonic language arises from the rather intuitive Karabali-Schnitzer arguments[46, 44].

However, the next indication is stronger, since it is based on real computations, and concerns the Schwinger model case. We generally find a confinement potential, which disappears for a particular θ-world and for massless fermions. Such a computation is done using the same semi-classical reasonning advocated afterwards in the non-abelian case. The full quantum theory subsequently confirms this picture, by means of the construction of exotic states, which only commutes with the mass term for the particular θ-world where screening prevails, confirming screening in that and in the massless cases. For a detailed discussion and abundant literature concerning those points, see chapter 10 of reference [1]. This settles the question in full for the Schwinger model.

In the non-abelian case, although not solvable from first principles, the situation is analogous. After finding the screening potential in eq. (3.139), with the various limits correctly

reproduced, we have been able to construct the state (3.142), which commutes with the mass term, being thus observable. This puts the non-abelian case in pair with the Schwinger model, since the operator described realizes the screening picture, albeit not being the most general quantum solution, which is however beyond the scope of the discussion, since for our aim it is not necessary to display the full set of exotic states, one single being enough to confirm the issue. One further confirmation of the result consists in noticing that the exotic state thus constructed is trivial in the abelian case, since the combination $\varphi + \eta$ acts as a constant there. Thus, the screening phenomenon is strictly non-abelian.

The generalization of the method to the 3+1 dimensional case requires a detailed knowledge of the bosonic form of the action, which as commented before, contains quantum information at the classical level. Therefore, although technically much more complicated, the method itself has chances of being applicable. Since we are dealing with static solutions of the equations of motion, the results are presumably at least as trustful as the instanton gas formulation of QCD.

One possible source of corrections is to consider the external charges as obeying a certain dynamics, being classical solutions of field equations[49]. The present methods as applied to that case do not differ in any fundamental way from what we presented here.

3.7 Integrability

3.7.1 Equations of motion and integrability condition

We are now going to deal with the action S_{eff} given in (3.64), obtaining further important information. Due to the presence of higher derivatives in that action, it is convenient to introduce an auxiliary field and rewrite it in the equivalent form (3.57), or else (3.58).

The equation of motion of the W-field is easily computed. The WZW contribution has been obtained in [41], and the Yang-Mills action leads to an extra term. We obtain

$$\begin{aligned}&\frac{c_V+1}{4\pi}\partial_+\left(W\partial_-W^{-1}\right)+\frac{1}{(4\pi\mu)^2}\partial_+\partial_-\left(W\partial_-W^{-1}\right)-\frac{1}{(4\pi\mu)^2}\partial_+[W\partial_-W^{-1},\partial_+\left(W\partial_-W^{-1}\right)]\equiv\\&\frac{c_V+1}{4\pi}\partial_+\left(W\partial_-W^{-1}\right)+\frac{1}{(4\pi\mu)^2}\partial_+\mathcal{D}_-\left(W\partial_-W^{-1}\right)=0\end{aligned}\tag{3.147}$$

We can list the relevant field operators appearing in the definition of the conservation law (3.147), that is

$$I_-^W=\frac{1}{4\pi}(c_V+1)J_-^W+\frac{1}{(4\pi\mu)^2}\partial_+\partial_-J_-^W-\frac{1}{(4\pi\mu)^2}[J_-^W,\partial_+J_-^W]\quad ,\tag{3.148}$$

with $\partial_+I_-^W=0$, and $J_-^W=W\partial_-W^{-1}$. It is straightforward to compute the Poisson algebra, using the canonical formalism, which in the bosonic formulation includes quantum corrections. We have

$$\left\{I_{ij}^W(t,x),I_{kl}^W(t,y)\right\}=\left[I_{kj}^W\delta_{il}-I_{il}^W\delta_{kj}\right]\delta(x^1-y^1)-\alpha\delta^{il}\delta^{kj}\delta'(x^1-y^1)\quad ,$$

$$\begin{aligned}\left\{I^W_{ij}(t,x), J^W_{-kl}(t,y)\right\}&=(J^W_{-kj}\delta_{il} - J^W_{-il}\delta_{kj})\delta(x^1-y^1) + 2\delta_{il}\delta_{kj}\delta'(x^1-y^1) \quad , \quad (3.149)\\ \left\{J^W_{ij}(t,x), J^W_{-kl}(t,y^1)\right\}&=0 \quad .\end{aligned}$$

where $\alpha = \frac{1}{2\pi}(c_V+1)$. We thus obtain a current algebra for I^W_-, acting on J^W_- with a central extension.

3.7.2 Dual case-non local formulation

At the Lagrangian level, we find the Euler-Lagrange equations for β from the perturbed WZW action (3.77), that is,

$$\delta\Gamma[\beta]=\left[\frac{1}{4\pi}\partial_-(\beta^{-1}\partial_+\beta)\right]\beta^{-1}\delta\beta \quad , \tag{3.150}$$

$$\delta\Delta(\beta)=2\Big(\partial_+^{-1}(\beta^{-1}\partial_+\beta) - [\partial_+^{-2}(\beta^{-1}\partial_+\beta),(\beta^{-1}\partial_+\beta)]\Big)\beta^{-1}\delta\beta \quad . \tag{3.151}$$

We define the current components

$$J^\beta_+=\beta^{-1}\partial_+\beta \quad , \tag{3.152}$$

$$J^\beta_-=-4\pi\mu^2\partial_+^{-2}J^\beta_+ = -4\pi\mu^2\partial_+^{-2}(\beta^{-1}\partial_+\beta) \quad , \tag{3.153}$$

which summarize the β equation of motion as a zero-curvature condition given by

$$[\mathcal{L},\mathcal{L}] = [\partial_+ + J^\beta_+, \partial_- + J^\beta_-] = \partial_- J^\beta_+ - \partial_+ J^\beta_- + [J^\beta_-, J^\beta_+] = 0 \quad . \tag{3.154}$$

This is not a Lax pair, as *e.g.* in the usual non-linear σ-models, where J^β_μ is a conserved current and a conserved non-local charge is obtained. However, to a certain extent, the situation is simpler in the present case, due to the rather unusual form of the currents, which permits us to write the commutator as a total derivative, in such a way that in terms of the current J^β_- we have

$$\partial_+\left(4\pi\mu^2 J^\beta_- + \partial_+\partial_- J^\beta_- + [J^\beta_-, \partial_+ J^\beta_-]\right) = 0 \quad . \tag{3.155}$$

Therefore the quantity

$$I^\beta_-(x^-) = 4\pi\mu^2 J^\beta_-(x^+,x^-) + \partial_+\partial_- J^\beta_-(x^+,x^-) + [J^\beta_-(x^+,x^-), \partial_+ J^\beta_-(x^+,x^-)] \tag{3.156}$$

does not depend on x^+, and it is a simple matter to derive an infinite number of conservation laws from the above.

Canonical quantization proceeds straightforwardly, and as a consequence we can compute the algebra of conserved currents, which is analogous to (3.149).

This means that two-dimensional QCD contains an integrable system[1, 50]. Moreover, it corresponds to an off-critical perturbation of the WZW action. If we write $\beta = \mathrm{e}^{i\phi} \sim 1 + i\phi$, we verify that the perturbing term corresponds to a mass term for ϕ. The next natural step is to obtain the algebra obeyed by (3.156), and its representation. However, there is a difficulty presented by the non-locality of the perturbation.

3.8 Algebraic aspects of QCD$_2$ and integrability

We saw that two-dimensional QCD, although not exactly soluble, in terms of free fields, is a theory from which some valuable results may be obtained. The $1/N$ expansion reveals a simple spectrum valid for weak coupling, while the strong coupling offers the possibility of understanding the baryon as a generalized sine-Gordon soliton. Moreover, the $1/N$ expansion of the pure-gauge case may be performed, and the partition function is equivalent to one of a string model described by a topological field theory, the Nambu-Goto string action, and presumably terms preventing folds.

All such results point to a relatively simple structure, which could be mirrored by an underlying symmetry algebra. In fact such algebraic structures do exist. In the above-mentioned case of the large-N expansion of pure QCD$_2$, one finds a W_∞-structure related to area-preserving diffeomorphisms of the Nambu-Goto action. A W_∞ structure for gauge-invariant bilinears in the Fermi fields can be constructed [51]. Such is an algebra which appears also in fermionic systems, and in the description of the quantum Hall effect. Moreover, pure QCD$_2$ is equivalent to the $c = 1$ matrix model, which also has a representation in terms of non-relativistic fermions, and contains a W_∞ algebra as well. The problem is also related to the Calogero-Sutherland models. The mass eigenstates build a representation of the W_∞ algebra as found in [51].

After bosonizing the theory, further algebraic functions of the fields turn out to obey non-trivial conservation laws, as we will see. The theory can be related to a product of several conformally invariant WZW sectors, a perturbed WZW sector, all related by means of BRST constraints, which play a very important role in gauge theories. A dual formulation exists and permits us to study the theory in both strong and weak coupling limits. Finally, once displayed, the relation to Calogero systems and further integrable models is also amenable to understanding in the previous framework.

3.9 Conclusions

In twenty years of development, two-dimensional QCD made outstanding contributions to the non-perturbative comprehension of strong interactions. The large-N limit of the theory revealed a desirable structure for the mesonic spectrum, whose higher levels display a Regge behaviour. These properties were generalized for fermions in the adjoint representation. This is an important step towards understanding the theory in higher dimensions, since adjoint matter can substitute the lack of the transverse degrees of freedom of the gauge field in two dimensions. Matter in the adjoint representation of the gauge group provides fields which mimic the transverse degrees of freedom characteristic of gauge theories in higher dimensions, and may show more realistic aspects of strong interactions. The main new point consists in the presence of a phase transition indicating a deconfining temperature. Further properties of the perturbative theory are also in accordance with expectations for strong interactions, and therefore it has certain advantages over the usual non-linear σ-models in the description of strong interactions by means of simplified models. The large-N limit of

QCD_2 is smooth and provides a picture of the string in the Feynman diagram space. In certain low-dimensional systems, the $1/N$ expansion turns out to be the correct expansion. In models with problematic infrared behaviour, such as $\mathbb{C}P^{N-1}$ and Gross-Neveu models, properties such as confinement and spontaneous mass generation are easily obtained in the large-N approximation and the S-matrices are explicitly checked[52].

The computation of the non-Abelian fermionic determinant is the key to tacking the problem of confinement which provides an effective theory for the description of the mesonic bound states, and to opening the possibility of understanding baryons as solitons of the effective interactions. Major problems of QCD can be dealt with using these methods.

The string interpretation of pure Yang-Mills theory, as well as its Landau-Ginzburg-type generalizations connects the previously mentioned picture to that of non-critical string theory. These developments form the basis for a deeper understanding of the rôle of non-critical string theory in the realm of strong interactions. String theory although far from being realized, seems to be the correct way to understand strong interactions at intermediate energies. For full details see [53].

The question of high-energy scattering in strong interactions is linked with two-dimensional integrable models. Thus the higher-symmetry algebras, spectrum-generating algebras, and integrability conditions, might give clues to the understanding of two-dimensional QCD. At high energies, Feynman diagrams simplify and become effectively two dimensional. The theory may be described in the impact parameter space, and in the case of QCD_4, the Reggeized particles scatter according to an integrable Hamiltonian.

In the massless case, the external field problem for the effective action in particular can be analyzed and the computation of gauge current and fermionic Green's functions can be reduced to the calculation of tree diagrams. These features are also not covered by 't Hooft's method. As an example, if we take the pseudo-divergence of the Maxwell equation, i.e.,

$$\left(\mathcal{D}^2 + (c_V + 1)\frac{e^2}{\pi}\right) F = 0 \quad , \tag{3.157}$$

where $F = \epsilon_{\mu\nu} F^{\mu\nu}$. This equation generalizes the analogous result obtained for the Schwinger model to the non-Abelian case. This suggests that an intrinsic Higgs mechanism, analogous to the one well-known in QED_2, can also characterize the non-Abelian theory. This is, nevertheless, not contained in 't Hooft's approach, since the mass arises from the fermion loop, which also contributes to the axial anomaly. This is suppressed in the $1/N$ expansion.

In spite of difficulties, QCD_2 has also served as a laboratory for gaining insight into various phenomenological aspects of four-dimensional strong interactions, such as the Brodsky-Farrar scaling law for hadronic form factors, the Drell-Yan-West relation or the Bloom-Gilman duality for deep inelastic lepton scattering.[54]

The next important step towards understanding this theory is its relation to string theory, or SCD_2. It concerns one of the most important applications of the theory of non-critical strings.[55]

The general problem of strong interactions did not progress substantially until recently as far as it concerns low-energy phenomena. Such a problem should be addressed using non-

perturbative methods, since perturbation theory of strong interactions is only appropriate for the high-energy domain, missing confinement, bound-state structure and related phenomena. In fact, several properties concerning hadrons are understandable by means of the concept of string-like flux tubes, which are consistent with linear confinement and Regge trajectories, as well as the approximate duality of hadronic scattering amplitudes, which are the usual concepts of the string idea. In fact, a similar idea is already present in the construction of the dipole of the Schwinger model, in which case it is, however, far too simple to be realistic.

In short, these ideas support the suggestion that the understanding of the theory of strong interactions requires the study of the large-N limit of QCD. Although there are several candidate models for such a theory in two dimensions, concerning a thorough comprehension of subtle problems such as confinement cannot be obtained without the inclusion of QCD_2.

Chapter 4

Two-dimensional gravity

4.1 Introduction

String theory is intensively discussed in modern quantum field theory. It has been under focus for the past thirty years, had a complex development, served different purposes and has undergone major changes.

The theory emerged from the dual models whose aim was to present an alternative to quantum field theory, one which would not rely on perturbative schemes. In particular, the failure of perturbative approach to strong interactions, was the main motivation of dual models and subsequently the initial objective of string theory. Due to the lack of a dynamical principle, the dual models were not predictive enough to be tested experimentally. Nevertheless many ideas fructified, as e.g. the concept of duality and the Veneziano formula, which later permitted a reconciliation of the dual models with quantum field theory by means of the introduction of the concept of string dynamics.

With the advent of supersymmetry and the GSO projection, string theory, which initially aimed at the explanation of strong interactions was then targeting a more ambitious task: the formulation of a unified theory of all interactions.

Thus, one sees already two aspects of the string models, the first concerns strong interactions where non-critical strings is used to describe the physical space-time which, in principle, cannot be studied by the usual string dynamics. In physical dimensions, string theory is anomalous and a Wess Zumino term has to be introduced to restore the symmetries. Non-critical strings incorporating the Wess-Zumino term are valuable for studying various statistical mechanical problems and have lead to the development of matrix models, which have applications in areas such as nuclear physics, and transport phenomena.

The second aspect of strings, that which presents it as a model of all interactions, still follows a very refined and mathematically sophisticated path. Today, theories of strings and superstrings are believed to contain higher symmetries which relate different types of strings. Such is the duality symmetry, which also lead to the re-newed interest on extended objects: the membrane (or more generally the p-brane) theories. Here, we shall concentrate on the problems related to two-dimensional gravity as a model, and partially as a tool for

the understanding of non-critical string theory, without getting into the specific details of string theory.

4.2 Polyakov formulation of the Nambu-Goto string action

The classical dynamics of strings is entirely based on geometry. The Nambu-Goto action is proportional to the area swept by the string as it evolves in space-time, that is

$$S_{NG} = \lambda \int \mathrm{d}^2\xi \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2} = \lambda \int \mathrm{d}^2\xi (\det \gamma_{ab})^{1/2} \quad , \tag{4.1}$$

where $X^\mu(\xi_0, \xi_1)$ describes the position of the string and

$$\gamma_{ab} = \partial_a X^\mu \partial_b X_\mu \tag{4.2}$$

is the induced metric.

This action describes a field X^μ which obeys the minimum area equation $\partial_a[(\det\gamma)^{1/2}\gamma^{ab}\partial_b x^\mu] = 0$. This equation is equivalent to the two-dimensional Klein-Gordon equation, supplemented by the so-called Virasoro conditions which is in turn obtained by requiring the metric to be diagonal. It is not difficult to see that this same set of conditions is obtained from a free field action for X^μ which incorporates a two-dimensional gravitational field $g_{\alpha\beta}$. Such is the Polyakov[56] string action

$$S = \frac{1}{2} \int \mathrm{d}^2\xi \sqrt{|g|} g^{ab} \partial_a X^\mu \partial_b X_\mu \quad , \tag{4.3}$$

where $\xi = (\xi^1, \xi^2)$ are local coordinates on $M, X^\mu(\xi), (\mu = 1, ..., D)$ defines the embedding from the world-sheet to the D-dimensional space-time, and $g = \det g_{ab}$. For reasons to become clear later on, D is taken to be arbitrary for the time being.

In the Polyakov formulation, the connection of string theory with two-dimensional gravity becomes most transparent. Indeed, M is a compact orientable two-dimensional manifold with boundary ∂M, and with metric tensor g_{ab}. In such a case, $X^\mu(\xi) : M \to \mathbb{R}^D$ is an embedding of M in D-dimensional Minkowski space. The classical Polyakov action is then given by (4.3).

4.3 The effective action of quantum gravity

We now proceed to discuss the quantization of the theory defined by the Polyakov action (4.3). The classical action is invariant under reparametrizations as well as under Weyl transformations. However at the quantum level only one of these symmetries can be maintained. This is analogous to the situation which arises in gauge theories, where either vector or axial vector current conservation can be implemented at the quantum level. We shall maintain

reparametrization invariance, at the expense of loosing Weyl invariance. Renormalization will in general require the introduction of counter terms, which in general complicate the structure of a simple diffeomorphism invariant action. Making use of this invariance, we proceed to quantize the theory in the conformal gauge. Taking into account the Faddeev-Popov determinant and after integrating over the fields X^μ of the "target" space, we arrive at an effective action of the "Liouville" type. This action has a coefficient which only vanishes in 26 dimensions. This shows that in the critical dimension Weyl invariance can also be implemented and consequently the dynamics is that of a free field. In $D \neq 26$, the breakdown of Weyl invariance leads to a non-trivial dynamics.

4.3.1 Uniqueness of the Polyakov action

At the classical level, the Polyakov action (4.3) is not the most general action compatible with all symmetries of the theory. The requirement of Poincaré invariance implies that the corresponding Lagrangian can only depend on the derivatives of X^μ, since the translations $X^\mu \to X^\mu + a^\mu$ must represent a symmetry of the action; the further requirement of diffeomorphism invariance permits the following generalized form of the action

$$S'[x,g]\frac{1}{2}A\int_M d^2\xi\sqrt{-g}g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{1}{2\pi}B\int_M d^2\xi\sqrt{-g}R + \mu^2\int d^2\xi\sqrt{-g} \tag{4.4}$$

where A, B and μ^2 play the role of renormalization constants, and R is the Ricci curvature scalar. We refer to (4.4) as the Polyakov-Zheltukin action.

The second and third terms in the action (4.4) are the Einstein action (without matter fields) and the cosmological term, respectively. The former does not contribute to the equations of motion, but merely modifies the boundary conditions (which we do not consider in this sequel). The equation of motion associated with the Einstein action

$$S_{Einstein} = \int d^2\xi\sqrt{-g}R \tag{4.5}$$

reads

$$R_{ab} - \frac{1}{2}g_{ab}R = 0 \tag{4.6}$$

and is trivially satisfied, since the two-dimensional geometry leads to the identity $R_{ab} \equiv \frac{1}{2}g_{ab}R$. The Gauss-Bonnet theorem states that Einstein's Lagrangian is a total derivative in two dimensions. This means that the two-dimensional action is a topological invariant quantity which measures the genus of the manifold on which one integrates. For a compact manifold M with b boundaries one has

$$S_{Einstein} = \int d^2\xi\sqrt{-g}R = 2\pi n = 2\pi(1-g) \tag{4.7}$$

where g is the genus of the manifold and the number $2(1-g)$ is its Euler characteristic. Since it is a topological invariant quantity, the Euler characteristic does not participate in the equation of motion, which turns out to be a constraint for the gravitational field (T_{ab}

(matter) = 0), rather than a dynamical equation. This is the case in string and superstring theories at the critical dimension (where also Weyl invariance holds). Indeed, the graviton (and gravitino) equations of motion ensure reparametrization (and super-reparametrization) invariance by means of the Virasoro (and super Virasoro) conditions.

4.3.2 Quantum Gravity

We shall now proceed with the quantization of the Polyakov model. Following Polyakov, we shall do this within the functional framework. Restricting ourselves for the time being to manifolds without boundaries and handles, we consider the expression

$$\mathcal{Z} = \int \mathcal{D}g \int \mathcal{D}X^{\mu} \mathrm{e}^{-\int d^2\xi \sqrt{g} g^{ab} \partial_a X^{\mu} \partial_b X_{\mu}} \quad . \tag{4.8}$$

For the Euclidean partition function associated with the Polyakov action $A = 2$, $B = \mu^2 = 0$). This is a merely formal expression, since as a result of the reparametrization invariance of the action, this functional integral is actually infinite. In order to define a finite integral we shall have to adopt the Faddeev-Popov procedure. At the classical level, Polyakov's action is proportional to the area of a two-dimensional surface on the manifold M. We are thus faced with the problem of summing over surfaces randomly immersed in a D-dimensional Euclidean space. The analogous problem of one-dimensional curves randomly immersed in the Euclidean space corresponds to a system of Brownian motion and to the quantum theory of a free relativistic point particle after continuation to Minkowski space. It is thus natural that in the case in question we are lead to the quantum theory of strings.

To begin with we need to define the measure in (4.8). Polyakov has discussed this question in detail.

Going to the conformal gauge

Since the action (4.4) is diffeomorphism invariant Faddeev-Popov procedure is required for the evaluation of the partition function.

The first step in this procedure is taken by choosing a gauge. In particular, on a sphere (no handles and no boundaries) we may always choose the conformal gauge in which the metric tensor takes the form

$$g_{ab}(\xi) = e^{\phi(\xi)} \eta_{ab} \quad . \tag{4.9}$$

This gauge can only be chosen if we perform a suitable conformal transformation. There are however topological obstructions.

Computation of The Faddeev-Popov determinant

We can complete the derivation of the effective action of quantum gravity by explicitly computing the Faddeev-Popov determinants appearing in the measure (4.8). The corresponding heat kernel is expected to have a de Witt-Seeley expansion. Combining the results of both

matter and ghost calculations, we arrive at the final form of the effective action, which is given by

$$S_{tot}^{eff} = \frac{26 - D}{48\pi} \int d^2\xi \left[\delta^{ab}\partial_a\phi\partial_b\phi + \mu^2 e^{-\phi}\right] \tag{4.10}$$

We shall refer to S_{eff} as the Liouville action.

4.4 The Liouville theory

The classical Polyakov action is invariant under reparametrization and Weyl transformations. Quantization breaks Weyl-invariance, but preserves conformal invariance. This allows one to choose the conformal gauge in order to simplify the calculations. After gauge fixing on the orbits defined by the reparametrization, the infinite group volume corresponding to the conformal transformations can be factored out at the expense of a Faddeed-Popov determinant represented in terms of a ghost action of the Liouville type. On the other hand, the one-loop effective action breaks the Weyl invariance, and hence shows a non-trivial dependence on the conformal factor in the metric. This dependence is once again, of the Liouville type. In the critical dimension $D = 26$, the one-loop contributions of the original action and the ghost action cancel out. Hence, a further gauge fixing is required, by means of which the group volume corresponding to the Weyl transformations can once more be factored out.

The situation witnessed here is in fact analogous to that occurring in anomalous chiral gauge theories in the gauge-invariant formulation. The chiral invariance of the effective one-loop action is restored in that case by the Wess-Zumino term. In the present case this rôle is taken up by the Liouville action associated with the Faddeev-Popov term of diffeomorphism invariance.

In the absence of the Weyl anomaly the partition function (4.8) of quantum gravity describes a purely topological theory which is characterized by the genus of the manifold. It is the breakdown of Weyl invariance in the non-critical dimension ($D \neq 26$) which adds a term of the form (4.10) to the action, thereby rendering the dynamics non-trivial.

4.4.1 The classical Liouville theory

The Liouville equation has been known for more than a century. In classical mathematics it has been used in order to study the uniformization problem by Poincaré. A knowledge of Liouville theory is essential for understanding this important mathematical problem.

The generalized Liouville action is

$$S = \frac{1}{4\pi} \int \mathrm{d}^2 z \sqrt{|\hat{g}|} \left\{ \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{2\mu}{\gamma^2}\mathrm{e}^{\gamma\phi} + \frac{1}{\gamma}\phi\hat{R}(\hat{g}) \right\} \quad , \tag{4.11}$$

where $\hat{g}$ is a fiducial metric and we have found it convenient to rescale the Liouville field as $\phi \to \gamma\phi$, with $\gamma = \sqrt{\frac{c}{6\pi}}$. The last term in (4.11) vanishes in flat space.

In the flat space ($\hat{R} = 0$), the equation of motion of the Liouville field has the general solution

$$\mathrm{e}^{\gamma\phi(z,\overline{z})} = \frac{4}{\mu}\frac{A'(z)B'(\overline{z})}{(1 - A(z)B(\overline{z}))^2} \quad , \tag{4.12}$$

where the prime indicates differentiation with respect to the arguments.

On the other hand, on the space of a punctured Riemann sphere it is always possible to find a conformal map such that the curvature is localized at isolated singular points. In this case, the curvature $\hat{R}(\hat{g})$ is given in terms of delta functions. Thus (4.12) continues to be a solution away from the singularities.

Let us consider the case of one such a singularity at the origin of the z-plane. Neglecting for the moment the cosmological term, the Liouville field obeys the equation,

$$\Delta\phi + \frac{1-a}{\gamma} 8\pi\delta^{(2)}(z) = 0 \quad , \tag{4.13}$$

with the solution

$$\phi = \frac{a-1}{\gamma} \ln (z\overline{z}) \quad , \tag{4.14}$$

where a parametrizes the strength of the curvature at the origin. It is essential to check that neglecting the cosmological term does not affect the above solution near the singularity. From (4.14) we have

$$\mathrm{e}^{\gamma\phi} \sim |z|^{-2(1-a)} \quad , \tag{4.15}$$

which is integrable only for $a > 0$. This forbids the existence of highly-curved single points. As a matter of fact, the results below show that the classical solution to the field equations exactly matches the above condition. This result will also be used to characterize the conditions to be fulfilled by the constants in the quantum theory.

Next we include the cosmological term and study the different solutions of the generalized Liouville equation (we only include one source of curvature)

$$\Delta\phi - \frac{2\mu}{\gamma}\mathrm{e}^{\gamma\phi} + \frac{1-a}{\gamma} 8\pi\delta^{(2)}(z) = 0 \quad , \tag{4.16}$$

and obtain solutions of the type (4.12) with $A(z) = z^a$ and $B(\overline{z}) = \overline{z}^a$,

$$\mathrm{e}^{\gamma\phi} = \frac{4}{\mu}\frac{a^2}{(z\bar{z})^{1-a}[1-(z\bar{z})^a]^2} \quad . \tag{4.17}$$

This concludes our summary of classical Liouville theory.

4.4.2 The quantum Liouville theory

Quantum Liouville theory has been studied intensively (see [55]). It has been discussed as a classical integrable model with boundary conditions [57], and the string spectrum has been analysed[58]. The full quantum operator solution has been studied in ref. [59] where

conformal invariance has been considered in detail. The model including boundary terms was also studied in detail[60].

The previous prolegomena warrant the importance of the Liouville theory in the study of two-dimensional random surfaces, in particular of quantum gravity. As we shall see, it also serves as a useful device for obtaining the exact correlation functions of the dressed vertex operator in non-critical string theory using the well known Coulomb gas method[61]. Moreover, one is able to partially treat[62] the Wheeler-de Witt equation[63], allowing for a better understanding of time in this simplified model, an ill defined concept in general relativity[64]. Furthermore it is possible to compare different approaches to non-critical string theory, namely Liouville continuous approach at one hand, and matrix models (discrete) approach on the other hand.

By integrating the equation of motion (4.16) over the whole space, using $R = -\gamma \mathrm{e}^{-\gamma\phi}\Delta\phi$, the Gauss-Bonnet theorem, and replacing $\frac{1-a}{\gamma}$ by various external sources of curvature of strength β_i, we obtain

$$\gamma \sum \beta_i + 2h - 2 - \frac{\mu}{8\pi} A = 0 \quad , \tag{4.18}$$

where h is the number of handles of the Riemann surface, and A is its area. This relation ensures that a classical solution exists if and only if

$$\gamma \sum \beta_i + 2h - 2 > 0 \quad . \tag{4.19}$$

To quantize the Liouville theory, we can use the canonical method. The energy momentum tensor is traceless, i.e. $T_{+-} = 0$, which implies conformal invariance. We define the physical Hilbert space of states by requiring the other components of the energy momentum tensor to be weakly zero. (The conformal dimension of the primary operator $\mathrm{e}^{\beta\varphi}$ can be fixed.)

The equation of motion obeyed by the exponential field can be shown to be equivalent to the definition of a null vector in conformal field theory; it can be written as[55]

$$\left(\partial_z^2 + \gamma^2 T(z)\right) \mathrm{e}^{-\frac{1}{2}\gamma\phi} = 0 \quad , \tag{4.20}$$

showing that $\mathrm{e}^{-\frac{1}{2}\gamma\phi}$ is a solution of a null vector equation, thereby making contact with the methods of conformal field theory.

In order to have a better understanding of Liouville theory, we shall consider the Schrödinger problem associated with the Hamiltonian of the Liouville action, so that we can obtain (4.20) from first principles.

Let us consider the Lagrangian

$$\mathcal{L} = \frac{\sqrt{|\hat{g}|}}{4\pi} \left(\frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu\phi \partial_\nu\phi + \frac{2\mu}{\gamma^2} \mathrm{e}^{\gamma\phi} - \frac{Q}{2} \phi \hat{R} \right) \quad , \tag{4.21}$$

where Q is classically related to the coupling constant γ by $Q = -\frac{2}{\gamma}$. This relation will however acquire quantum corrections. The $(++)$ component of the energy momentum tensor

defines a Hamiltonian as

$$H = \frac{1}{2}(\phi' + 4\pi P)^2 + Q(\phi' + 4\pi P)' + \frac{\mu}{2\gamma^2}\mathrm{e}^{\gamma\phi} + \frac{Q^2}{16}\hat{R} + \frac{Q^2}{8} \quad , \tag{4.22}$$

where P is the momentum canonically conjugated to ϕ. Note that this Hamiltonian is the generator of conformal transformations. For flat space $\hat{R} = 0$. In the so-called minisuperspace approximation, where the quantities do not depend on the space variable x^1, the corresponding eigenvalue problem in the Schrödinger representation reads

$$H\psi \equiv \left(-\frac{1}{2}\frac{\partial^2}{\partial\phi^2} + \frac{\mu}{2\gamma^2}\mathrm{e}^{\gamma\phi} + \frac{Q^2}{8}\right)\psi = \Delta\psi \quad , \tag{4.23}$$

where Δ is the conformal dimension of ψ. Since the potential vanishes exponentially for $\phi \to -\infty$, we have for the eigenfunctions of H

$$\psi\,(\phi \to -\infty) \approx \sin p\phi \tag{4.24}$$

where p is the eigenvalue of $-i\frac{\partial}{\partial\phi}$, and $\Delta = \frac{1}{2}p^2 + \frac{1}{8}Q^2$. However, there also exist eigen solutions corresponding to vertex operators of the form $\mathcal{O} \sim \mathrm{e}^{\beta\phi}$. These are not normalizable, but are important in the context of non-critical string theory. Solving the Schrödinger Eq. (4.23) we see that they are given by

$$\psi_{\mathcal{O}}(\phi) = \mathrm{e}^{(\frac{1}{2}Q+\beta)\phi} \tag{4.25}$$

with conformal dimension

$$\Delta = -\frac{1}{2}(\beta + \frac{Q}{2})^2 + \frac{Q^2}{8} \tag{4.26}$$

This is the general case in Liouville field theory as discussed by Seiberg[65].

Let us now consider the "puncture" operator, obtained as a solution of Eq. (4.23) with $\Delta = Q^2/8$. There are two solutions: $\psi = 1$ and $\psi = \phi$, corresponding to the operators $\mathrm{e}^{\frac{Q}{2}\phi}$ and $\phi\mathrm{e}^{\frac{Q}{2}\phi}$ respectively.

An equation of motion of the form (4.16) may be obtained from the study of correlation functions of the type

$$\left\langle \prod_i \mathrm{e}^{\beta_i\phi(z_i)} \right\rangle = \int \mathcal{D}\phi \mathrm{e}^{-S[\phi]} \prod_i \mathrm{e}^{\beta_i\phi(z_i)} \quad . \tag{4.27}$$

Therefore, in the quantum theory it is important to analyze the action of the operators $\mathrm{e}^{\beta\phi(z)}$ on the vacuum; these operators are responsible for the local curvature strength β. That is to say that they generate elliptic solutions such as (4.17), with $a = 1 - \gamma\beta$. From the discussion following (4.17) we see that we need $\beta\gamma < 1$, or [1] $\beta > 1/\gamma$. Classically this is equivalent to requiring the inequality

$$\beta \geq -\frac{Q}{2} \quad , \tag{4.28}$$

[1] $\gamma < 0$, this is the case in the quantum theory, according to our conventions

which continues to be correct even after taking into account quantum corrections. Canonical quantization of the theory has been performed in [66], and one finds for the anomalous dimension of the above vertex operators

$$\Delta(\mathrm{e}^{\beta\phi}) = -\frac{1}{2}\beta(\beta+Q) = -\frac{1}{2}(\beta+\frac{Q}{2})^2 + \frac{c-1}{24}, \tag{4.29}$$

where $c = 1 + 3Q^2$ is the central charge of the Virasoro operator. Note that (4.29) is in agreement with (4.26) obtained in the minisuperspace approximation. The result (4.29) reflects the fact that Liouville theory cannot be treated as a free theory. In fact, Liouville theory cannot be treated in perturbation theory due to the lack of a normalizable ground state at finite values of the Liouville field [2].

A comparison between the full action of string theory and the Liouville field, shows that the latter can be thought of as a target space coordinate, and the full action is that of a string theory in a non-trivial background. Thus, in quantum theory we shall sum over the possible geometries. We consider the Liouville interaction with curvature and recall that in string theory the sum over geometries corresponds to a string perturbation theory in terms of the genus, that is

$$Z \sim \sum Z_h g_{st}^{2\chi} \quad . \tag{4.30}$$

In addition, we use eq. (4.21) for constant Liouville configurations and take into account the contribution from the fiducial metric to the Euler characteristic of the manifold. Collecting these results together we find that the string coupling constant is related to the Liouville field by

$$g_{st} = g_0 \mathrm{e}^{-\frac{Q}{2}\phi} \quad . \tag{4.31}$$

The importance of this result will be shown later when we define the relation between the tachyon vertex and the corresponding wave function, for example in the discussion concerning eq. (4.25).

4.4.3 Canonical quantization and $SL(2,R)$ symmetry

Liouville theory corresponds to two-dimensional gravity in the conformal gauge. In order to obtain the corresponding formulation in the light-cone-gauge, we notice that the reparametrization invariant action ($\Box$ is the Laplace-Beltrami operator)

$$S_g = \frac{c}{96\pi}\int \mathrm{d}^2x[\sqrt{|g|}R\frac{1}{\Box}R + \mu^2\sqrt{|g|}] \quad , \tag{4.32}$$

reduces to the Liouville action in the conformal gauge.

In order to study the symmetries of (4.32) it is convenient to use a formulation in terms of a local Lagrangian. We thus introduce an auxiliary field φ[1]

$$\varphi = -\frac{\alpha}{2}\Box^{-1}R \quad , \tag{4.33}$$

[2] for $\gamma\phi \to -\infty$ all derivatives drop to zero, and the theory is trivial[67]

where $\alpha^2 = 8\kappa$. Thus classically we have

$$\Box\varphi + \frac{1}{2}\alpha R = 0 \tag{4.34}$$

and

$$\kappa R \frac{1}{\Box} R = -\frac{1}{2}(\varphi\Box\varphi + \alpha R\varphi) \quad . \tag{4.35}$$

We obtain correlators and operator-product expansions for the elementary fields of the theory. The supersymmetric extension is also amenable to the method, and analogous expressions have been obtained in the literature. See [1] for an extensive literature.

Chapter 5

Four-dimensional analogies and consequences

It is of prime importance to generalize the concept of quasi-integrability to higher dimensions. Indeed, Bardeen [32] has recently pointed out that helicity amplitudes in high-energy QCD are very simple at tree level and are described by a self-dual Yang-Mills theory. The classical solution of this theory strongly resembles the Bethe Ansatz solution of integrable two-dimensional models. Moreover, the one-loop amplitudes are reminiscent of those corresponding to anomalous conservation laws. It is known that the self-dual Yang-Mills theory is an integrable theory and is described by very simple actions [68]. On the other hand, integrable Lagrangeans with either anomalies [23] or with non-vanishing amplitudes for particle production[69] are known and are well documented in the literature. It remains an interesting open problem to see whether the quasi-integrability idea is the most efficient framework for the description of non-trivial dynamics in theories with higher conservation laws, in general space-time dimensions, in spite of the Coleman-Mandula no-go theorem [70] and its more general version [71].

The self-dual Yang-Mills equations can be solved in two different ways. The first method uses the zero curvature condition for $A_\pm$ which are solved by means of the introduction of group valued elements, g and h, leading to the definitions

$$-\frac{ie}{\sqrt{2}}A_{0+z} = g^{-1}\partial_{0+z} \quad , \quad -\frac{ie}{\sqrt{2}}A_{x-iy} = g^{-1}\partial_{x-iy}g \quad ,$$
$$-\frac{ie}{\sqrt{2}}A_{0-z} = h^{-1}\partial_{0-z}h \quad , \quad -\frac{ie}{\sqrt{2}}A_{x+iy} = h^{-1}\partial_{x+iy}h \quad .$$

We can write the remaining information in terms of $H = gh^{1-}$, obtaining for H the equation

$$\partial_{0-z}(H^{-1}\partial_{0+z}H) - \partial_{x+iy}(H^{-1}\partial_{x-iy}H) = 0\,, \tag{5.1}$$

which is the Euler-Lagrange equation of an action proposed by Donaldson, and by Nair and Schiff[72], and which reads

$$S[H]{=}\frac{f_\pi^2}{2}\int d^4x\,\mathrm{tr}\left(\partial_{0+z}H\partial_{0-z}H^{-1} - \partial_{x-iy}H\partial_{x+iy}H^{-1}\right)$$

$$+\frac{f_\pi^2}{2}\int d^4x dt \,\mathrm{tr}\Big([H^{-1}\partial_{0+z}H, H^{-1}\partial_{0-z}H]- \\ [H^{-1}\partial_{x-iy}H, H^{-1}\partial_{x+iy}H]\Big)\, H^{-1}\partial_t H \,. \tag{5.2}$$

On the other hand, if one interchanges equations of motion and gauge prepotential definitions, using

$$A_{x+iy} = 0 \quad , \quad A_{0+z} = \sqrt{2}\partial_{x+iy}\Phi \quad , \quad A_{x-iy} = \sqrt{2}\,\partial_{0-z}\Phi \quad , \tag{5.3}$$

one obtains the Leznov action[73]

$$S_{Leznov} = f_\pi^2 \int d^4x \,\mathrm{tr}\left(\frac{1}{2}\partial\phi\cdot\partial\phi + \frac{ie}{3}\phi\,[\partial_{x+iy}\phi, \partial_{0-z}\phi]\right) \tag{5.4}$$

whose equation of motion is equivalent to the remaining equation for self-dual Yang-Mills, that is

$$\partial^2\Phi - ie\,[\partial_{x+iy}\Phi, \partial_{0-z}\Phi] = 0 \tag{5.5}$$

There is no proof of equivalence of both (namely DNS and Leznov action), but there are hints in this direction. It is the interactive solution of Leznov equation which is of the type of series predicted by the Bethe Ansatz approach to two-dimensional integrable models. The detailed analysis of Cangemi[74] leads to that direction in a clear way.

Chapter 6

Conclusions and Final Remarks

Two-dimensional quantum field theory provides a very powerful laboratory for obtaining insight into the non-perturbative aspects of quantum field theory. The kinematical simplifications which occur in two dimensional space-time have allowed for the complete solution of a variety of models involving interacting fields. The non trivial nature of these solutions provide a deeper insight into the structure of quantum field theory, and has found useful applications in several areas of research, such as string theories and systems of statistical mechanics at criticality.

In all cases of completely solvable models, free bosonic fields and their exponentials, as well as the boson-fermion equivalence for free fields play a key role in the explicit construction of the correlation functions. The extension of these results to the case of massive fermions takes us, strictly speaking, outside the realm of free fields. Nevertheless, by doing an expansion in the mass parameter, the equivalence of the massive Thirring model to the sine Gordon equation could be proven on the basis of zero mass bosonic fields alone. Mandelstam's representation of fermions in terms of bosons showed that the fermion of the massive Thirring model could be identified with the soliton of the sine Gordon theory. The existence of such soliton sectors provides a link with the order disorder algebra in statistical mechanics. The existence of an infinite number of conservation laws at the classical and quantum level, eventually allowed the exact S matrix in both the soliton-soliton sector and soliton-bound-state sector, to be constructed. These ideas prove useful in more algebraic approaches to QFT.

The study of two dimensional gauge theories provides a deeper insight into features which are believed to be shared by four dimensional chromodynamics. Among these properties are the θ-vacuum, the screening of "colour" and confinement of quarks, the Nambu-Goldstone realization of the η'-the so called $U(1)$ problem- and the topology of the configuration space over which one functionally integrates.

Conservation laws, $1/N$ expansion, S-matrix factorization and operator techniques, were essential tools for the non perturbative treatment of the $O(N)$ and $SU(N)$ chiral Gross-Neveu models, non linear sigma models and $\mathbb{C}P^{N-1}$ models. These models turned out to be much more complex than a $U(1)$ gauge theory of fermions interacting with gauge fields via

minimal coupling. In the case of sigma models, a very rich structure emerged. For this theory, geometry was the main guideline, since following the ideas developed in the context of gravity, one wished to formulate a theory in terms of the largest possible number of symmetries. This was discussed within the framework of differential geometry at the classical level. The outcome was a very elegant description of integrable theories, allowing for a Lax pair at the classical level, thus generalizing the ideas developed for the sine Gordon theory. At the quantum level, one obtains field theories with exactly computable S-matrices. Integrability properties of these sigma models are found to hold also for a classical supersymmetric Yang-Mills theory and supergravity in 10 dimensions, as well as for the respective theories obtained by dimensional reduction in 4 dimensions. This opens the possibility for non-trivial models in higher dimensions.

One of the realistic applications of sigma models, is in the realm of string theories and quantum gravity, where the background field method can be used to calculate the quantum corrections to the Einstein's equations. The requirement of conformal invariance in the framework of string theory, enforces the vanishing of the sigma model β function, which turns out to be the "string corrected" gravitational equations of motion.

The construction of non-trivial S-matrices on the basis of factorization, and its relation to an infinite number of conservation laws, has useful applications in algebraic quantum field theory. In fact, conformally invariant theories are known to possess a rich algebraic structure derived from the fusion rules of the elementary fields. These fusion rules being associative, and at the same time non-trivial, imply certain algebraic relations known from the mathematical literature as Artin Braid relations. These relations are similar to the factorization relations obtained for factorizable S-matrices. A general structure commonly known as "quantum groups", or to be more precise, "quantum algebras", emerge naturally from these constructions.

The anomalous two dimensional chiral gauge theories provide a useful laboratory for understanding the role played by gauge anomalies in chiral gauge theories. In this respect, chiral QCD_4 shares many properties with two dimensional QCD. In fact, in any attempt to understand string theories away from criticality, or to deal with the Weinberg-Salam model of the weak interactions one must deal with the problem of chiral anomalies.

One of the most promising applications of two-dimensional models to the description of fundamental interactions is that of string theory. Strings describe a two-dimensional world-sheet embedded in the target manifold, in which the string moves. The dynamic is thus governed by two dimensional quantum field theory. Because of reparametrization invariance of string theory, conformal invariance can be extensively used to understand the string. In fact, interaction of strings is formulated by using vertex operators, which are generalizations of the exponential fields.

The old string model, as well as the Neveu-Schwarz-Ramond model, could be described by a two dimensional local Lagrangian. The discovery of space-time supersymmetry by means of the Gliozzi-Scherk-Olive construction, permitted one to envisage possible applications to grand unified theories, since supersymmetry was a requirement to solve the hierarchy problem in unified theories; this use of string theories had been proposed earlier, from the

interpretation of the spin 2 (lowest) state of closed string theory, as the graviton, and strings would in such a case be a theory of quantum gravity. This unification attempt was later enhanced by important works on the cancellation of anomalies. This led to a new superstring theory in which left and right movers were treated differently, i.e., the heterotic string.

Of course, our ultimate objective is to understand the dynamics of the real world. Many successes of quantum field theory merely rely on kinematical arguments, such as the idea of dynamical symmetry breaking and the construction of representations of the gauge groups in the matter sector. The actual non-perturbative dynamics of quantum field theory in four dimensions remains largely unknown. Hence, applications of the experiences gained from the study of two dimensional QFT, to higher dimensions is highly sought and scores success. It is rewarding that the techniques of two dimensional QFT provides many of the expected results, sometimes just "taken for granted" in higher dimensions, due to the lack of methods for proving or disproving their validity. We have however also witnessed some surprising features which are non-perturbative in nature and are difficult to be understood in a perturbative context.

Nevertheless, some important results concerning generalizations to higher dimensions have already been obtained. Bosonization of fermions is also possible in three dimensional space-time if we have a gauge field with a Chern-Simons density in the Lagrangian. Three dimensional models have been studied in this context. The ideas developed in the context of two dimensional quantum field theory thus appear to represent a step in the right direction. Moreover, further recent progress made in the study of random surfaces, shows that the ideas of string theory have a wider range of validity. In fact it is thought that a phase transition in string theory occurs, such that the theory can accommodate the grand unification ideas, statistical models, and strong interactions, in a common framework.

More recently, direct use of the integrability ideas in order to establish a computational procedure for the high energy limit of non-abelian gauge theory has been advocated. The results seem promising, and one envisages the possibility of describing the gluon interactions in terms of a two-dimensional integrable spin system. Moreover, it is also possible to describe the vast complications of perturbative amplitudes of Yang-Mills theories in terms of a single component field described by a very simple action connected with self-dual Yang-Mills theory, which is by far simpler than the original theory itself.

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